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PRIME IDEALS IN STRUCTURAL MATRIX NEAR-RINGS

ENOCH K. S. LEE

Abstract. This paper studies different types of prime ideals and their radicals in structural matrix near-rings. Relationships between various types of prime ideals of a near-ring and the corresponding structural matrix near-ring are given.

1. Introduction

Matrix near-rings were first studied by Heatherly [5] and Ligh [6] in the context of distributive and n-distributive near-rings. In 1986, Meldrum and van der Walt [9] defined a matrix near-ring over a near-ring as a subnear-ring of a transformation near-ring. Certain prime ideals and their associated radicals have been studied since then. (See Booth and Groenewald [1] and Groenewald [4].) Furthermore, van der Walt and van Wyk [11] initiated the study of structural martix near-rings (see also [12]). A structural matrix near-ring "$\mathcal{M}_n(B,R)$", which depends virtually on the structure of the Boolean matrix "B", is a subnear-ring of the matrix near-ring "$\mathcal{M}_n(R)$". In this paper, we study several types of prime ideals and their radicals in structural matrix near-rings over a near-ring with identity . We also provide a different way of describing the set $(R^n(j,L):R^n(j,R))$, which was studied in [11], [12], and [13]. For basic properties concerning near-rings and matrix near-rings, we refer to [3], [7], [9], and [10].

2. Preliminaries

Throughout this paper, a near-ring will be a right zero-symmetric near-ring with identity, while a subnear-ring will always be a subnear-ring with the identity. By an ideal of a near-ring we will mean a two-sided ideal. R will be a generic symbol for a near-ring (except where noted). Let n be a fixed natural number and let $\underline{n}$ denote the set $\{1,2,\ldots,n\}$. R^n will denote the direct sum of n copies of $(R,+)$, and elements of R^n will

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be represented by $\overline{u}, \overline{v}$, etc., and considered as n-tuples, for instance, $\overline{u} = (u_1, \cdots, u_n)$. If $\overline{u}, \overline{v} \in R^n$, we define the product $\overline{u}\ \overline{v} = (u_1v_1, \cdots, u_nv_n)$; if also $a \in R$, let $\overline{u}a = (u_1a, \cdots, u_na)$. Denote the n-tuple with 1 in the i-th component and 0 elsewhere by $\overline{\epsilon}_i$.

Recall that $\mathcal{M}_0(G)$ is the near-ring of all zero-preserving mappings of $(G, +)$ into itself for any group $(G, +)$. The mappings ι_j and π_j will denote the j-th coordinate injection and projection, respectively. That is $\iota_j(u) = (0, \cdots, u, \cdots, 0)$ where u is in the j-th position and $\pi_j(u_1, \ldots, u_n) = u_j$. For $r \in R$, let $f^r \in M_0(R)$ be defined by $f^r(s) = rs$ for all $s \in R$. We define $f_{ij}^r : R^n \to R^n$ where $f_{ij}^r = \iota_i f^r \pi_j$ for $i, j \in \underline{n}$ and $r \in R$.

Definition 2.1 [9] The subnear-ring of $M_0(R^n)$ generated by the set $\{f_{ij}^r | r \in R; i, j \in \underline{n}\}$ is called an $n \times n$ *matrix near-ring* over R, denoted by $\mathcal{M}_n(R)$, and each element of $\mathcal{M}_n(R)$ is called a *matrix*.

An $n \times n$ matrix $B = [b_{ij}]$ will be a generic symbol for a *Boolean matrix* of size n, i.e., $b_{ij} = 0$ or 1 for $i, j \in \underline{n}$. We shall henceforth assume the Boolean matrix B satisfies the conditions: (1) $b_{ii} = 1$ for each $i \in \underline{n}$; and (2) if $b_{ij} = b_{jk} = 1$, then $b_{ik} = 1$. Let $\overline{u},\ \overline{v} \in R^n$. We wirte:

$$\overline{u} \sim_i \overline{v} \text{ if and only if } \pi_j\overline{u} = \pi_j\overline{v} \text{ for all } j \text{ such that } b_{ij} = 1.$$

Remark. If $b_{ij} = 0$, then $\overline{\epsilon}_j \sim_i \overline{0}$ where $\overline{0} = (0, \cdots, 0)$.

Definition 2.2. [11] Let $\mathcal{M}_n(B, R) = \{X \in \mathcal{M}_n(R) | (\forall_i \in \underline{n}, \forall \overline{u}, \overline{v} \in R^n)(\overline{u} \sim_i \overline{v} \Rightarrow \pi_i X\overline{u} = \pi_i X\overline{v})\}$. We call $\mathcal{M}_n(B, R)$ the $n \times n$ structural matrix near-ring over R with respect to B.

Van der Walt and van Wyk [11] showed that $\mathcal{M}_n(B, R)$ is also the subnear-ring of $\mathcal{M}_n(R)$ generated by the set $\{f_{ij}^r | r \in R \text{ and } b_{ij} = 1\}$. In view of this result, we can now introduce the concept of representations of structural matrices. Let $\mathbb{E}_n(B, R)$ be the subset of words over the alphabet of symbols $\{f_{ij}^r | r \in R \text{ and } b_{ij} = 1\} \cup \{(,), +\}$, recursively defined by the following rules:

(1) $f_{ij}^r \in \mathbb{E}_n(B, R)$ for all $r \in R$ and for $i, j \in \underline{n}$ with $b_{ij} = 1$;
(2) if $\mathbf{X}$ and $\mathbf{Y} \in \mathbb{E}_n(B, R)$, then $\mathbf{X} + \mathbf{Y} \in \mathbb{E}_n(B, R)$;
(3) if $\mathbf{X}$ and $\mathbf{Y} \in \mathbb{E}_n(B, R)$, then $(\mathbf{X})(\mathbf{Y}) \in \mathbb{E}_n(B, R)$.

The *length* of an element $\mathbf{X}$ of $\mathbb{E}_n(B, R)$ is defined as the number of f_{ij}^r in $\mathbf{X}$ such that $b_{ij} = 1$. (Note: there might be more than one expression for any matrices. The "length" is defined on expressions of matrices, not on matrices.) The *weight*, $\omega(A)$, of a matrix A of $\mathcal{M}_n(B, R)$, is the length of an expression in $\mathbb{E}_n(B, R)$ of minimal length representing A. Observe that for any matrix A of $\mathcal{M}_n(B, R)$ with $1 < \omega(A)$, there exist matrices $C, D \in \mathcal{M}_n(B, R)$ of minimal length representing A. Observe that for any matrix A of $\mathcal{M}_n(B, R)$ with $1 \leq \omega(C), \omega(D) < \omega(A)$ such that $A = C + D$ or CD.

The following results will be useful throughout this paper.

Lemma 2.3.

(1) [*Lemma* 2.1, 1] *If* $X \in \mathcal{M}_n(R)$, $s \in R$ *and* $\overline{u} \in R^n$, *then* $X(\overline{u}s) = (X\overline{u})s$.

(2) [*Proposition* 2.2, 11] *Let* $X \in \mathcal{M}_n(B,R)$. *For any* $\overline{u}$ *and* $\overline{v} \in R^n$, *if* $\overline{u} \sim_i \overline{v}$, *then* $X\overline{u} \sim_i X\overline{v}$.

(3) [*Lemma* 2.3, 11] *Let* $i, j \in \underline{n}$. *Then* $b_{ij} = 1$ *if and only if* $f_{ij}^r \in \mathcal{M}_n(B,R)$ *for all* $r \in R$.

Definition 2.4. Let $\mathcal{L} \subseteq \mathcal{M}_n(B,R)$ and $j \in \underline{n}$. Then:

(1) $\prod_B(R,j) = \{(u_1,\cdots,u_n) \in R^n | u_i = 0 \text{ if } b_{ij} = 0\}$.

(2) $\mathcal{L}_{(j)}[B] = \{x \in R \mid (\exists X \in \mathcal{L})(\exists \overline{u} \in \prod_B(R,j))(x = \pi_j X\overline{u})\}$.

If there is no ambiguity, we write $\Pi(R,j)$ and $\mathcal{L}_{(j)}$ for $\prod_B(R,j)$ and $\mathcal{L}_{(j)}[B]$, respectively.

Remark.

(1) If $X \in \mathcal{M}_n(B,R)$ and $i,j \in \underline{n}$, then $f_{ii}^1 X f_{jj}^1 = f_{ij}^x$ where $x = \pi_i X \overline{\epsilon}_j$.

(2) $(\mathcal{M}_n(B,R))_{(i)} = R$ for $i \in \underline{n}$.

Suppose H is a subset of R such that $RH \subseteq H$. Then we say H is a *left invariant subset*. *Right invariant* and *two-sided invariant subsets* can be defined in a similar way.

Lemma 2.5.

(1) *Let* $\overline{u} = (u_1,\cdots,u_n) \in \prod_B(R,j)$. *Then* $\overline{u} = (\sum_{i=1}^n f_{ij}^{u_i})\overline{\epsilon}_j$.

(2) *Let* $\mathcal{Y}$ *be a right invariant subset of* $\mathcal{M}_n(B,R)$ *and* $j \in \underline{n}$. *If* $X \in \mathcal{Y}$ *and* $\overline{u} \in \prod_B(R,j)$, *then there is* $Y \in \mathcal{Y}$ *such that* $X\overline{u} = Y\overline{\epsilon}_j$.

Proof. Since $(u_1,\cdots,u_n) \in \prod(R,j)$, if $b_{ij} = 0$ then $u_i = 0$. Therefore we have part (1). Part (2) follows immediately from part (1) and the fact that $\overline{\epsilon}_j \in \prod(R,j)$.

Proposition 2.6. *Let* $\mathcal{L}$ *be a two-sided invariant subset of* $\mathcal{M}_n(B,R)$ *and* $i \in \underline{n}$. *Then* $x \in \mathcal{L}_{(i)}$ *if and only if* $f_{ii}^x \in \mathcal{L}$. *Moreover, if also* $b_{ij} = b_{ji} = 1$, *then* $\mathcal{L}_{(i)} = \mathcal{L}_{(j)}$.

Proof. Note that $x \in \mathcal{L}_{(i)}$ if and only if $x = \pi_i X\overline{u}$ for some $X \in \mathcal{L}$ and $\overline{u} \in \prod(R,i)$. By Lemma 2.5(2), there is a $Y \in \mathcal{L}$ such that $X\overline{u} = Y\overline{\epsilon}_i$ and so $x = \pi_i Y\overline{\epsilon}_i$. Thus $f_{ii}^x = f_{ii}^1 Y f_{ii}^1 \in \mathcal{L}$. Conversely, from the fact that $\overline{\epsilon}_i \in \prod(R,i)$ and $x = \pi_i f_{ii}^x \overline{\epsilon}_i$, we have: if $f_{ii}^x \in \mathcal{L}$, then $x \in \mathcal{L}_{(i)}$.

Suppose also that $b_{ij} = b_{ji} = 1$. Since $f_{ii}^x \in \mathcal{L}$ if and only if $f_{jj}^x = f_{ji}^1 f_{ii}^x f_{ij}^1 \in \mathcal{L}$, we then have the desired result.

Proposition 2.7. *Let* $\mathcal{L}$ *be an ideal of* $\mathcal{M}_n(B,R)$. *Then* $\mathcal{L}_{(i)}$ *is an ideal of* R *for any* $i \in \underline{n}$.

Proof. Suppose $x, y \in \mathcal{L}_{(i)}$ and $r, s \in R$. We have $f_{ii}^{x-y} = f_{ii}^{x} - f_{ii}^{y}$ is in $\mathcal{L}$. Hence $x - y$ is in $\mathcal{L}_{(i)}$. Using a similar argument, we can show $r + x - r$, xr, $s(x+r) - sr$ are in $\mathcal{L}_{(i)}$. Therefore $\mathcal{L}_{(i)}$ is an ideal of R.

Observe that whenever $\mathcal{L}$ is an ideal of $\mathcal{M}_n(B, R)$, we have: $\mathcal{L}$ is a proper ideal if and only if $\mathcal{L}_{(i)}$ is a proper ideal for some $i \in \underline{n}$.

Proposition 2.8. *If $j \in \underline{n}$ and $X \in \mathcal{M}_n(B, R)$, then $X(\prod_B(R, j)) \subseteq \prod_B(R, j)$.*

Proof. In view of Lemma 2.5, we note that it suffices to show that $\pi_i X \overline{\epsilon}_j = 0$ whenever $X \in \mathcal{M}_n(B, R)$ and $b_{ij} = 0$. But this follows immediately from the fact that $\overline{\epsilon}_j \sim_i \overline{0}$ whenever $b_{ij} = 0$. (See the remark after Definition 2.1.)

Definition 2.9. Let $i \in \underline{n}$ and $L \subseteq R$. Then:
(1) $\coprod_B(i, L) = \{(u_1, \cdots, u_n) \in R^n | u_j \in L \text{ if } b_{ij} = 1\}$;
(2) $L^{(i)}[B] = \{X \in \mathcal{M}_n(B, R) | X(\prod_B(R, i)) \subseteq \coprod_B(i, L)\}$.

If no confusion can occur, we write $\coprod(i, L)$ and $L^{(i)}$ for $\coprod_B(i, L)$ and $L^{(i)}[B]$, respectively.

Remark.
(1) Let $X \in \mathcal{M}_n(B, R)$ and $L \subseteq R$. We have $X \in L^{(i)}$ if and only if $\pi_j X \overline{u} \in L$ whenever $\overline{u} \in \prod(R, i)$ and $b_{ij} = 1$.
(2) $R^{(i)} = \mathcal{M}_n(B, R)$ for any $i \in \underline{n}$.

Lemma 2.10. *let L be a left ideal of R and $i \in \underline{n}$. Then $\coprod_B(i, L)$ is an $\mathcal{M}_n(B, R)$-ideal of R^n. (Here we consider R^n as an $\mathcal{M}_n(B, R)$-module.)*

Proof. Obviously $\coprod(i, L)$ is a normal subgroup of R^n. We prove $X(\overline{u} + \overline{v}) - X\overline{v}$ is in $\coprod(i, L)$ for $\overline{u} \in \coprod(i, L)$, $\overline{v} \in R^n$, $X \in \mathcal{M}_n(B, R)$ by means of induction on the weights of matrices. In fact, it suffices to show $\pi_j(X(\overline{u} + \overline{v}) - X\overline{v}) \in L$ for all j such that $b_{ij} = 1$. So assume $b_{ij} = 1$. If $\omega(X) = 1$, then $X = f_{hk}^{r}$ with $b_{hk} = 1$. We have:

$$\pi_j(f_{hk}^{r}(\overline{u} + \overline{v}) - f_{hk}^{r}\overline{v}) = \begin{cases} 0 & \text{if } j \neq h, \\ r(u_k + v_k) - rv_k & \text{if } j = h. \end{cases}$$

If $j = h$, then $b_{ik} = b_{ij}b_{jk} = 1$ and hence $u_k \in L$. Therefore we have $\pi_j(f_{hk}^{r}(\overline{u} + \overline{v}) - f_{hk}^{r}\overline{v}) \in L$. For purposes of induction, we assume $X(\overline{u} + \overline{v}) - X\overline{v}$ is in $\coprod(i, L)$ for $X \in \mathcal{M}_n(B, R)$ with $1 \leq \omega(X) \leq m$. Now if $\omega(X) = m + 1$, then there are $C, D \in \mathcal{M}_n(B, R)$ with $1 \leq \omega(C), \omega(D) \leq m$ such that either $X = C + D$ or CD. In the first case we have $X(\overline{u} + \overline{v}) - X\overline{v} = C(\overline{u} + \overline{v}) + (D(\overline{u} + \overline{v}) - D\overline{v}) - C\overline{v}$ is in $\coprod(i, L)$. In the second case we have $X(\overline{u} + \overline{v}) - X\overline{v} = C((D(\overline{u} + \overline{v}) - D\overline{v}) + D\overline{v}) - CD\overline{v}$ is in $\coprod(i, L)$. By the principle of induction, we have that $\coprod(i, L)$ is an $\mathcal{M}_n(B, R)$-ideal of R^n.

Observe that if we assume that L is a left R-subgroup in the above lemma, then we can show that $X(\coprod_B(i, L)) \subseteq \coprod_B(i, L)$ for $X \in \mathcal{M}_n(B, R)$.

Proposition 2.11.

(1) *If L is an ideal of R, then $L^{(i)}$ is an ideal of $\mathcal{M}_n(B,R)$ for $i \in \underline{n}$.*
(2) *If L is a left R-subgroup, then $L^{(i)}$ is a two-sided $\mathcal{M}_n(B,R)$-subgroup for $i \in \underline{n}$.*

Proof. Proposition 2.8 and Lemma 2.10 give part (1). Use proposition 2.8 and the observation after Lemma 2.10 to obtain part (2).

Lemma 2.12.
(1) *Let L be a right invariant subset of R. Then $f_{ii}^x \in L^{(i)}$ if and only if $x \in L$.*
(2) *Let L be a proper two-sided R-subgroup and $b_{kh} = 1$. Then $f_{kh}^1 \in L^{(i)}$ if and only if $b_{ik} = 0$ or $b_{hi} = 0$.*

Proof.
(1) Suppose $f_{ii}^x \in L^{(i)}$. Since $\overline{\epsilon}_i \in \prod(R,i)$, we have $f_{ii}^x\overline{\epsilon}_i \in \coprod(i,L)$. In particular, $x = \pi_i f_{ii}^x \overline{\epsilon}_i \in L$. Suppose now $x \in L$, $\overline{u} = (u_1,\cdots,u_n) \in \prod(R,i)$, and $b_{ih} = 1$. Then:
$$\pi_h f_{ii}^x \overline{u} = \begin{cases} 0 & \text{if } h \neq i, \\ xu_i & \text{if } h = i. \end{cases}$$
So $\pi_h f_{ii}^x\overline{u} \in L$. This yields $f_{ii}^x\overline{u} \in \coprod(i,L)$ and hence $f_{ii}^x \in L^{(i)}$.
(2) Assume $f_{kh}^1 \in L^{(i)}$. For purposes of contradiction, suppose $b_{ik} = b_{hi} = 1$. Thus f_{ik}^1, $f_{hi}^1 \in \mathcal{M}_n(B,R)$. This implies $f_{ii}^1 = f_{ik}^1 f_{kh}^1 f_{hi}^1 \in L^{(i)}$; hence $1 \in L$ from part (1). So L is not proper. This proves the result one way. Conversely, assume $b_{ik} = 0$ or $b_{hi} = 0$. Let $\overline{u} \in \prod(R,i)$ and $b_{im} = 1$. Then:
$$\pi_m f_{kh}^1 \overline{u} = \begin{cases} 0 & \text{if } m \neq k, \\ u_h & \text{if } m = k. \end{cases}$$
If $m = k$, then $b_{ik} = 1$. Thus from the assumption, we have $b_{hi} = 0$. Since $\overline{u} \in \prod(R,i)$, we have $u_h = 0$. (See Definition 2.4.) Then $\pi_m f_{kh}^1 \overline{u} = 0 \in L$. This yields $f_{kh}^1\overline{u} \in \coprod(i,L)$. Hence $f_{kh}^1 \in L^{(i)}$.

Proposition 2.13.
(1) *Let L be a proper two-sided R-subgroup and $k \in \underline{n}$. Then we have:*
$$(L^{(i)})_{(k)} = \begin{cases} R & \text{if } b_{ik} = 0 \text{ or } b_{ki} = 0, \\ L & \text{if } b_{ik} = b_{ki} = 1. \end{cases}$$
(2) *Let $\mathcal{L}$ be a left invariant subset of $\mathcal{M}_n(B,R)$. Then $\mathcal{L} \subseteq (\mathcal{L}_{(j)})^{(j)}$ for $j \in \underline{n}$.*

Proof.
(1) Note that $L^{(i)}$ and $(L^{(i)})_{(k)}$ are two-sided R-subgroups. (See Propositions 2.11(2) and 2.6.) Suppose $b_{ik} = 0$ or $b_{ki} = 0$. Lemma 2.12(2) gives $f_{kk}^1 \in L^{(i)}$. From Proposition 2.6, we have $1 \in (L^{(i)})_{(k)}$. Therefore $(L^{(i)})_{(k)} = R$. Suppose now $b_{ik} = b_{ki} = 1$. Proposition 2.6 implies $(L^{(i)})_{(i)} = (L^{(i)})_{(k)}$. Note $x \in (L^{(i)})_{(i)}$ if and only if $f_{ii}^x \in L^{(i)}$ if and only if $x \in L$. Hence $(L^{(i)})_{(k)} = L$.

(2) Observe that if $\overline{u} \in \prod(R, j)$ and $A \in \mathcal{L}$, then $\pi_j A\overline{u} \in \mathcal{L}_{(j)}$. So if $b_{ji} = 1$ and $X \in \mathcal{L}$, then $f_{ji}^1 \in \mathcal{M}_n(B, R)$ and hence $f_{ji}^1 X \in \mathcal{L}$ for $X \in \mathcal{L}$. This implies that whenever $\overline{u} \in \coprod(R, j)$ we have $\pi_i X\overline{u} = \pi_j f_{ji}^1 X\overline{u} \in \mathcal{L}_{(j)}$. This yields $X\overline{u} \in \coprod(j, \mathcal{L}_{(j)})$. Hence $X \in (\mathcal{L}_{(j)})^{(j)}$.

Lemma 2.14.
(1) *Let $U \subseteq V \subseteq R$. Then $U^{(i)} \subseteq V^{(i)}$ for $i \in \underline{n}$.*
(2) *Let $\mathcal{L} \subseteq \mathcal{K} \subseteq \mathcal{M}_n(B, R)$. Then $\mathcal{L}_{(i)} \subseteq \mathcal{K}_{(i)}$ for $i \in \underline{n}$.*
(3) *Let Γ be a collection of subsets of R. Then $(\cap_{U\in\Gamma} U)^{(i)} = \cap_{U\in\Gamma} U^{(i)}$ for $i \in \underline{n}$.*
(4) *Let Ω be a collection of two-sided invariant subsets of $\mathcal{M}_n(B, R)$. Then $(\cap_{\mathcal{L}\in\Omega} \mathcal{L})_{(i)} = \cap_{\mathcal{L}\in\Omega} \mathcal{L}_{(i)}$ for $i \in \underline{n}$. Furthermore, if $\mathcal{L}, \mathcal{K} \in \Omega$, then $\mathcal{L}_{(i)}\mathcal{K}_{(i)} \subseteq (\mathcal{L}\mathcal{K})_{(i)}$ for $i \in \underline{n}$.*

Proof. Parts (1), (2), and (3) follow immediately from definitions. We only show part (4). Use Proposition 2.6 to obtain that $x \in (\cap_{\mathcal{L}\in\Omega} \mathcal{L})_{(i)}$ if and only if $f_{ii}^x \in \cap_{\mathcal{L}\in\Omega} \mathcal{L}$ if and only if $x \in \mathcal{L}_{(i)}$ for $\mathcal{L} \in \Omega$. Thus we have $(\cap_{\mathcal{L}\in\Omega} \mathcal{L})_{(i)} = \cap_{\mathcal{L}\in\Omega} \mathcal{L}_{(i)}$. Furthermore, suppose $\mathcal{L}, \mathcal{K} \in \Omega$. If $x \in \mathcal{L}_{(i)}$ and $y \in \mathcal{K}_{(i)}$, then $f_{ii}^x \in \mathcal{L}$ and $f_{ii}^y \in \mathcal{K}$. Since $f_{ii}^{xy} = f_{ii}^x f_{ii}^y \in \mathcal{L}\mathcal{K}$, we have $xy \in (\mathcal{L}\mathcal{K})_{(i)}$.

Lemma 2.15. *Let L and H be proper two-sided R-subgroups. Then $L^{(i)} = H^{(j)}$ if and only if $L = H$ and $b_{ij} = b_{ji} = 1$.*

Proof. Suppose $L = H$ and $b_{ij} = b_{ji} = 1$. Observe that $\prod(R, i) = \prod(R, j)$ and $\coprod(i, L) = \coprod(j, L)$ will suffice to show $L^{(i)} = H^{(j)}$. If $\overline{u} \in \prod(R, i)$ and $b_{kj} = 0$, then $b_{ki} = 0$ and so $\pi_k\overline{u} = 0$. Thus $\overline{u} \in \prod(R, j)$ and hence $\prod(R, i) \subseteq \prod(R, j)$. Similarly, we have $\prod(R, j) \subseteq \prod(R, i)$. Therefore $\prod(R, i) = \prod(R, j)$. If $\overline{v} \in \coprod(i, L)$ and $b_{jk} = 1$, then $b_{ik} = 1$ and so $\pi_k\overline{v} \in L$. Thus $\overline{v} \in \coprod(j, L)$. This gives $\coprod(i, L) \subseteq \coprod(j, L)$. We then have $\coprod(i, L) = \coprod(j, L)$. This proves the result one way. Conversely, suppose $L^{(i)} = H^{(j)}$. From Proposition 2.13, we have:

$$L = (L^{(i)})_{(i)} = (H^{(j)})_{(i)} = \begin{cases} N & \text{if } b_{ij} = 0 \text{ or } b_{ji} = 0, \\ H & \text{if } b_{ij} = b_{ji} = 1. \end{cases}$$

This yields $L = H$ and $b_{ij} = b_{ji} = 1$.

3. Prime Ideals And Radicals

Recall that a proper ideal P of R is called
(1) a prime ideal if for any ideals U and V of R such that $(UV \subseteq P) \Rightarrow (U \subseteq P)$ or $(V \subseteq P)$.
(2) a 1-prime ideal if for any a, $b \in R$ such that $(aRb \subseteq P) \Rightarrow (a \in P)$ or $(b \in P)$.
(3) an equiprime ideal if for any $a \in R\backslash P$, x and $y \in R$ such that $(\forall r \in R, arx - ary \in P) \Rightarrow (x - y) \in P$. (See [1] and [2].)

(4) a completely prime ideal if for any a, $b \in R$ such that $(ab \in P) \Rightarrow (a \in P)$ or $(b \in P)$.

We write $\mathbf{P}_\nu(R)$ and $\mathbf{Spec}_\nu(R)$ for the intersection and the collection of all proper prime, 1-prime, equiprime, or completely prime ideals of R according to whether $\nu = 0, 1, e$, or 2.

Theorem 3.1. *Let P be a prime (resp. 1-prime, equiprime) ideal of R. Then $P^{(i)}$ is a prime (resp. 1-prime, equiprime) ideal of $\mathcal{M}_n(B,R)$ for $i \in \underline{n}$.*

Proof. Let P be a prime ideal of R and $\mathcal{U}$, $\mathcal{V}$ ideals of $\mathcal{M}_n(B,R)$ such that $\mathcal{UV} \subseteq P^{(i)}$. We want to show $\mathcal{U} \subseteq P^{(i)}$ or $\mathcal{V} \subseteq P^{(i)}$. Use Proposition 2.13 and Lemma 2.14 to obtain the following sequentially: $(\mathcal{UV})_{(i)} \subseteq (P^{(i)})_{(i)}$, $\mathcal{U}_{(i)}\mathcal{V}_{(i)} \subseteq P$, $\mathcal{U}_{(i)} \subseteq P$ or $\mathcal{V}_{(i)} \subseteq P$, and $\mathcal{U} \subseteq P^{(i)}$ or $\mathcal{V} \subseteq P^{(i)}$. Thus we are done. Now suppose P is a 1-prime ideal of R and X, $Y \in \mathcal{M}_n(B,R)$ such that X, $Y \notin P^{(i)}$. We want to show $X\mathcal{M}_n(B,R)Y \not\subseteq P^{(i)}$. From definitions, there are $\overline{u}$, $\overline{v} \in \prod(R,i)$ and h, $k \in \underline{n}$ with $b_{ih} = b_{ik} = 1$ such that $a = \pi_h X\overline{u} \notin P$ and $b = \pi_k Y\overline{v} \notin P$. Therefore there exists $r \in R$ such that $arb \notin P$. Since $\pi_h((X\overline{u})rb) = (\pi_h X\overline{u})rb = arb \notin P$, we have $(X\overline{u})rb \notin \coprod(i,P)$. Furthermore, $(X\overline{u})rb = X(\overline{u}rb)$ by Lemma 2.3 (1), so $X(\overline{u}rb) \notin \coprod(i,P)$. Observe that $\overline{u}rb = (u_1 r, \cdots, u_n r)(\pi_k Y\overline{v}) = (\sum_{j=1}^n f_{jk}^{u_j r})Y\overline{v}$ where $\overline{u} = (u_1, \cdots, u_n)$. Since $X(\sum_{j=1}^n f_{jk}^{u_j r})Y\overline{v} = X(\overline{u}rb) \notin \coprod(i,P)$, we have $X\mathcal{M}_n(B,R)Y \not\subseteq P^{(i)}$. Hence $P^{(i)}$ is 1-prime. The proof of the equiprime case is similar to that of the 1-prime case. (See also [Proposition 2.2, 1].)

Theorem 3.2. *Let $\mathcal{Q}$ be a 1-prime (resp. equiprime, completely prime) ideal of $\mathcal{M}_n(B,R)$. Then $\mathcal{Q}_{(i)}$ is a 1-prime (resp. equiprime, completely prime) ideal of R for $i \in \underline{n}$.*

Proof. We will prove the 1-prime case. Suppose $\mathcal{Q}$ is a 1-prime ideal of $\mathcal{M}_n(B,R)$. Let a and $b \in R$ such that $aRb \subseteq \mathcal{Q}_{(i)}$. Then $f_{ii}^{arb} \in \mathcal{Q}$ for all $r \in R$. Now if $X \in \mathcal{M}_n(B,R)$ and $x = \pi_i X\overline{\epsilon}_i$ then $f_{ii}^a X f_{ii}^b = f_{ii}^{axb} \in \mathcal{Q}$. This implies $f_{ii}^a \in \mathcal{Q}$ or $f_{ii}^b \in \mathcal{Q}$. Hence $a \in \mathcal{Q}_{(i)}$ or $b \in \mathcal{Q}_{(i)}$. Similarly, we can prove the equiprime and completely prime cases.

Lemma 3.3. *Let $\mathcal{Q}$ be an ideal of $\mathcal{M}_n(B,R)$ and $A \in \mathcal{M}_n(B,R)$. Then the following are equivalent:*

(1) $A \in (\mathcal{Q}_{(i)})^{(i)}$;

(2) If $\overline{u} \in \prod(R,i)$ and $b_{ih} = 1$, then $\pi_h A\overline{u} \in \mathcal{Q}_{(i)}$;

(3) If $\overline{u} \in \prod(R,i)$ and $b_{ih} = 1$, then $f_{ii}^{a_h} \in \mathcal{Q}$ where $a_h = \pi_h A\overline{u}$.

Proof. The equivalence of (1) and (2) follows directly from Definitions 2.4 and 2.9. The equivalence of (2) and (3) is obtained by using Proposition 2.6.

Theorem 3.4. *Let $\mathcal{Q}$ be a 1-prime ideal of $\mathcal{M}_n(B,R)$. Then there exists a $k \in \underline{n}$ such that $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. Hence $\cap_{i=1}^n (\mathcal{Q}_{(i)})^{(i)} = \mathcal{Q}$.*

Proof. We have shown that $\mathcal{Q} \subseteq (\mathcal{Q}_{(i)})^{(i)}$ for any $i \in \underline{n}$. Now suppose $X \notin \mathcal{Q}$. Since $(f_{11}^1 + \cdots + f_{nn}^1)X = X \notin \mathcal{Q}$, there exists $k \in \underline{n}$ such that $f_{kk}^1 X \notin \mathcal{Q}$. Furthermore from the fact that $\mathcal{Q}$ is 1-prime, we can find $T \in \mathcal{M}_n(B,R)$ such that $f_{kk}^1 XT f_{kk}^1 X \notin \mathcal{Q}$. This implies $f_{kk}^1 XT f_{kk}^1 \notin \mathcal{Q}$. Note that $f_{kk}^t = f_{kk}^1 XT f_{kk}^1 \notin \mathcal{Q}$ where $t = \pi_k XT \overline{\epsilon}_k$. Apply the preceding lemma to obtain $XT \notin (\mathcal{Q}_{(k)})^{(k)}$. So $X \notin (\mathcal{Q}_{(k)})^{(k)}$. This yields $(\mathcal{Q}_{(k)})^{(k)} = \mathcal{Q}$. The last part is now an immediate consequence.

Observe that in the above proposition if also $b_{jk} = b_{kj} = 1$, then $(\mathcal{Q}_{(j)})^{(j)} = \mathcal{Q}$. (See Proposition 2.6 and Lemma 2.15.) Theorems 3.1, 3.2, and 3.4 lead to our next results:

Theorem 3.5.
(1) $\mathbf{Spec}_0(\mathcal{M}_n(B,R)) \supseteq \{P^{(i)} | P \in \mathbf{Spec}_0(R) \quad \text{for } i \in \underline{n}\}$.
(2) $\mathbf{Spec}_v(\mathcal{M}_n(B,R)) = \{P^{(i)} | P \in \mathbf{Spec}_v(R) \quad \text{for } i \in \underline{n}\}$ *for* $v = 1, e$.

Theorem 3.6. $\mathbf{P}_\nu(\mathcal{M}_n(B,R)) = \cap_{i=1}^n (\mathbf{P}_\nu(R))^{(i)}$ *for* $\nu = 1$ or e.

To end this section, we study the cardinalities of $\mathbf{Spec}_\nu(\mathcal{M}_n(B,R))$ for $\nu = 0, 1, e$. But first we let $|W|$ be the cardinal of W for any set W.

Definition 3.7 Let $\approx$ be a relation on $\underline{n}$ (with respect to B) defined via:

$$i \approx j \text{ if and only if } b_{ij} = b_{ji} = 1.$$

Obviously, $\approx$ is an equivalence relation on $\underline{n}$. For convenience, denote by β the number of the equivalence classes induced by $\approx$ on $\underline{n}$. For instance, if B is an upper triangular matrix, then $i \approx j$ if and only if $i = j$. Hence $\beta = n$.

Theorem 3.8.
(1) $|\mathbf{Spec}_0(\mathcal{M}_n(B,R))| \geq |\mathbf{Spec}_0(R)| \cdot \beta$.
(2) $|\mathbf{Spec}_\nu(\mathcal{M}_n(B,R))| = |\mathbf{Spec}_\nu(R)| \cdot \beta$ for $\nu = 1$ or e.

Proof. See Lemma 2.15 and Theorem 3.5.

4. Concluding Remarks

In [11], van der Walt and van Wyk defined the set $R^n(j,L)$ to be:
$\{\overline{u} \in R^n | u_k = 0 \text{ if } b_{jk} = 1 \text{ and } b_{kj} = 0, \text{ and } u_k \in L \text{ if } b_{jk} = b_{kj} = 1\}$,
where $L \subseteq R$ and $j \in \underline{n}$ (see also [12] and [13]). They investigated the set $(R^n(j,L) : R^n(j,R))$ and proved that:

$$J_2(\mathcal{M}_n(B,R)) = \cap_{j=1}^n (R^n(j, J_2(R)) : R^n(j,R)).$$

We show that $(R^n(j,L) : R^n(j,R))$ coincides with $L^{(j)}$. This implies that we obtain a description of, for example, $\mathbf{P}_1(\mathcal{M}_n(B,R))$ analogous to the description of $J_2(\mathcal{M}_n(B,R))$ mentioned above. Without loss of generality, assume $0 \in L$, otherwise both $(R^n(j,L) :$

$R^n(j,R))$ and $L^{(j)}$ are empty. We need the following observations, the proofs are immediate from definitions:
(1) $R^n(j,L) \subseteq \coprod(j,L)$;
(2) $\prod(R,j) \subseteq R^n(j,R)$;
(3) $R^n(j,L) = R^n(j,R) \cap \coprod(j,L)$.
Use (1) and (2) to obtain $(R^n(j,L) : R^n(j,R)) \subseteq (\coprod(j,L) : \prod(R,j)) = L^{(j)}$.

Assume $X \in L^{(j)}$. To complete the proof, we need to show that $X(R^n(j,R)) \subseteq R^n(j,L)$. However, from part (3) and the fact that $R^n(j,R)$ is an $\mathcal{M}_n(B,R)$-ideal of R^n [Corollary 3.6, 11], it suffices to show $X(R^n(j,R)) \subseteq \coprod(j,L)$. This is equivalent to showing that $\pi_k X\overline{u} \in L$ whenever $\overline{u} \in R^n(j,R)$ and $b_{jk} = 1$. Assume $\overline{u} = (u_1, \cdots, u_n) \in R^n(j,R)$ and $b_{jk} = 1$. Furthermore if we could find an element $\overline{v}$ of $\prod(R,j)$ such that $\overline{u} \sim_k \overline{v}$, then Lemma 2.3 (2) yields $X\overline{u} \sim_k X\overline{v}$ and so $\pi_k X\overline{u} = \pi_k X\overline{v} \in L$. (Since $X \in L^{(j)}$ and $\overline{v} \in \prod(R,j)$, we have $X\overline{v} \in \coprod(j,L)$.)

Let $\overline{v} = (v_1, \cdots, v_n) \in R^n$ such that $v_m = u_m$ if $b_{mj} = b_{jm} = 1$, and $v_m = 0$ otherwise. Thus $\overline{v}$ is an element of $\prod(R,j)$. Suppose $b_{km} = 1$. We then have $b_{jm} = 1$ (since $b_{jk} = 1$). Therefore there are two possible cases: $b_{mj} = 1$ or $b_{mj} = 0$. If $b_{mj} = 1$, then $v_m = u_m$. If $b_{mj} = 0$, then $v_m = 0$ and $u_m = 0$ (since $b_{jm} = 1$ and $b_{mj} = 0$). This implies $\overline{u} \sim_k \overline{v}$. We are done.

Veldsman [14] used an example (of a finite near-ring R) given by Meldrum and Meyer [8] to show that $\mathbf{P}_0(\mathcal{M}_n(R))$ could be strictly contained in $(\mathbf{P}_0(R))^*$. (Note that $\mathcal{M}_n(R) = \mathcal{M}_n(B,R)$ where $B = [b_{ij}]$ with $b_{ij} = 1$ and $(\mathbf{P}_0(R))^* = (\mathbf{P}_0(R))^{(k)}$ for any $k \in \underline{n}$.) It would be interesting to determine the prime radical and all prime ideals of any matrix near-ring (or structural matrix near-ring).

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Mathematics Department, University of Southwestern Louisiana, Lafayette, LA 70504, USA.