

OSCILLATION THEOREMS FOR NONLINEAR SECOND ORDER DIFFERENCE EQUATIONS WITH A NONLINEAR DAMPING TERM

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Abstract. Some new sufficient conditions for the oscillation of the nonlinear damped second order difference equations of the form

$$\Delta^2 y_n + p_n \phi(y_n, \Delta y_n) + q_n f(y_{n+1})g(\Delta y_n) = 0$$

are given.

1. Introduction

In this paper we are concerned with the second order non-linear damped difference equation

$$\Delta^2 y_n + p_n \phi(y_n, \Delta y_n) + q_n f(y_{n+1})g(\Delta y_n) = 0, \quad n \in Z \quad (1)$$

where $Z = \{1, 2, \dots\}$, $\{p_n\}$ and $\{q_n\}$ are real sequences, the functions f and g are defined on the set R of real numbers, the function ϕ is defined on $R \times R$ and Δ is the forward difference operator, defined by $\Delta y_n = y_{n+1} - y_n$. It is interesting to study second order nonlinear difference equations because they have physical applications as evidenced by [1].

Throughout the paper the term solution of (1) is always used for a sequence $\{y_n\}$ which satisfies (1) for all $n \in Z$ and for which $\sup \{|y_n| : n \geq s\} > 0$ for any $s \in Z$. We make the standing hypothesis that equation (1) does possess such solutions. A solution is called non-oscillatory if it is eventually of constant sign. Otherwise it is called oscillatory.

Recently there has been an increasing interest in the study of the qualitative behaviour of solutions of difference equations of type (1) and /or related equations, see for example [1, 4, 5, 7-10] and the references cited therein.

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The purpose of this paper is to present theorems that give sufficient conditions for all solutions of (1) to be oscillatory. The obtained results are discrete analogues of some known theorems for non-linear differential equations due to Baker [2], Grace and Lalli [3] and Naito [6].

Now, for the difference equation (1) each result we shall prove, requires some of the following conditions.

- (c₁) $p_n \geq 0$ and $q_n > 0$ for all $n \in Z$;
(c₂) there exists a constant $M > 0$ such that

$$0 < \phi(x, y)y \leq My^2 \text{ for all } y \in R/\{0\} \text{ and } x \in R;$$

- (c₃) $yf(y) > 0$ and $g(y) > 0$ for all $y \neq 0$;
(c₄) f is non-decreasing and

$$f(-xy) \geq f(xy) \geq K_1 f(x)f(y) \text{ for } x, y > 0 \text{ for some constant } K_1 > 0;$$

- (c₅) g is non-increasing and

$$g(-xy) \geq g(xy) \geq K_2 g(x)g(y) \text{ for } x, y > 0 \text{ for some constant } K_2 > 0;$$

- (c₆) $f(y)g(y)/y^\alpha \geq \beta > 0$ for $y \neq 0$ where α is the ratio of odd positive integers.

In the sequel we need the following two Lemmas of which the first Lemma is due to Hooker and Patula [4] and the second Lemma is a discrete analogue of Baker [2].

Lemma 1. *If $y_N \geq 0$, $\Delta^2 y_n \leq 0$ and $\Delta y_n > 0$ for $n \geq N$, then $y_{n+1} \geq \frac{n}{2} \Delta y_n$ for all $n \geq 2N$.*

Lemma 2. *Suppose (c₁) – (c₃) hold. Further assume*

$$1 - Mp_n > 0 \text{ for all } n \in N \tag{2}$$

and

$$\sum_{n=N}^{\infty} \left(\prod_{s=N}^{n-1} (1 - Mp_s) \right) = \infty. \tag{3}$$

Then, if $\{y_n\}$ is a non-oscillatory solution of (1), we have $y_n \Delta y_n > 0$ for all sufficiently large $n \geq N$.

Proof. Let $\{y_n\}$ be a non-oscillatory solution of equation (1) and assume $y_n > 0$ for $n \geq n_0 \in Z$. Suppose $\{\Delta y_n\}$ is oscillatory, then there exists an integer $n_1 \geq n_0 \geq 0$ such that $\Delta y_{n_1} < 0$ or $\Delta y_{n_1} = 0$. First we consider $\Delta y_{n_1} < 0$. Now, from equation (1)

$$\Delta y_{n_1} \Delta^2 y_{n_1} + p_{n_1} \phi(y_{n_1}, \Delta y_{n_1}) \Delta y_{n_1} = -q_{n_1} f(y_{n_1+1}) g(\Delta y_{n_1}) \Delta y_{n_1} > 0$$

or

$$\begin{aligned}\Delta y_{n_1} \Delta^2 y_{n_1} &> -p_{n_1} \phi(y_{n_1}, \Delta y_{n_1}) \Delta y_{n_1} \\ \Delta y_{n_1} (\Delta y_{n_1+1} - \Delta y_{n_1}) &> -p_{n_1} M (\Delta y_{n_1})^2\end{aligned}$$

and hence

$$\Delta y_{n_1} \Delta y_{n_1+1} > (1 - Mp_{n_1}) (\Delta y_{n_1})^2 > 0.$$

Dividing the above inequality by the negative term Δy_{n_1} , we obtain

$$\Delta y_{n_1+1} < 0.$$

By induction, we obtain

$$\Delta y_n < 0 \text{ for all } n \geq n_1 \in Z.$$

Suppose $\Delta y_{n_1} = 0$, then equation (1) implies

$$\Delta y_{n_1+1} < 0,$$

and we obtain as above

$$\Delta y_n < 0 \text{ for all } n > n_1 \in Z.$$

Hence, in both the cases, we obtain

$$\Delta y_n < 0 \text{ for all } n \geq n_1$$

which however contradicts the assumption that $\{\Delta y_n\}$ oscillates. Thus $\{\Delta y_n\}$ is of fixed sign.

Let $\Delta y_n < 0$ for all $n \geq N \in Z$.

From (1) and (c_2) , we obtain,

$$\Delta u_n + Mp_n u_n \geq 0 \tag{4}$$

where $u_n = -\Delta y_n$. From (4) we have

$$u_n \geq u_N \prod_{s=N}^{n-1} (1 - Mp_s)$$

or

$$\Delta y_n \leq -u_N \prod_{s=N}^{n-1} (1 - Mp_s).$$

Summing the last inequality from N to $(n - 1)$, we obtain

$$y_n \leq y_N - u_N \sum_{s=N}^{n-1} \prod_{t=N}^{s-1} (1 - Mp_t).$$

In view of (3), we get

$$y_n \rightarrow -\infty \text{ as } n \rightarrow \infty, \text{ a contradiction.}$$

Remark. Note that when $p_n = 0$, conditions (2) and (3) are automatically satisfied.

2. Main Results

In this section we establish sufficient conditions for the oscillation of all solutions of equation (1). We begin with the following theorem.

Theorem 1. *In addition to conditions $(c_1) - (c_4)$, (2) and (3), suppose $f(u)g(u)$ is non-decreasing for $u \neq 0$,* (5)

$$\int_{+0} \frac{du}{f(u)g(u)} < \infty \text{ and } \int_{-0} \frac{du}{f(u)g(u)} < \infty \quad (6)$$

and

$$\sum_{n=0}^{\infty} q_n f(n) = \infty. \quad (7)$$

Then every solution of equation (1) is oscillatory.

Proof. Suppose that $\{y_n\}$ is a non-oscillatory solution of (1). In view of Lemma 2, there is no loss in generality in assuming that there is an integer N such that

$y_n > 0$ and $\Delta y_n > 0$ for all $n \geq N$. Since this implies that $\phi(y_n, \Delta y_n) > 0$ for all $n \geq N$, it follows that

$$\Delta^2 y_n + q_n f(y_{n+1})g(\Delta y_n) \leq 0. \quad (8)$$

Since f is non-decreasing, we have from Lemma 1

$$f(y_{n+1}) \geq f\left(\frac{n}{2}\Delta y_n\right) \text{ for all } n \geq 2N. \quad (9)$$

Using (9) in (8) and then applying condition (c_4) we obtain

$$\Delta^2 y_n + K_1^2 f\left(\frac{1}{2}\right) q_n f(n) f(\Delta y_n) g(\Delta y_n) \leq 0$$

or

$$\frac{\Delta^2 y_n}{f(\Delta y_n)g(\Delta y_n)} + K_1^2 f\left(\frac{1}{2}\right) q_n f(n) \leq 0 \text{ for all } n \geq 2N. \quad (10)$$

Observe $\Delta y_n \geq x \geq \Delta y_{n+1}$, we have

$$f(x)g(x) \leq f(\Delta y_n)g(\Delta y_n)$$

and therefore

$$\frac{\Delta^2 y_n}{f(x)g(x)} \leq \frac{\Delta^2 y_n}{f(\Delta y_n)g(\Delta y_n)}.$$

Using the above inequality in (10) and summing from $2N$ to n , we obtain

$$\begin{aligned} K_1^2 f\left(\frac{1}{2}\right) \sum_{s=2N}^n q_s f(s) &\leq \sum_{s=2N}^{n-1} \int_{\Delta y_{s+1}}^{\Delta y_s} \frac{dx}{f(x)g(x)} \\ &\leq \int_{y_n}^{y_{2N}} \frac{dx}{f(x)g(x)} < \infty \end{aligned}$$

which contradicts (7) and this proves the theorem.

Remark. In Theorem 1,

- a) if $p_n = 0$, then it reduces to Theorem 3 of [10],
- b) if $p_n = 0$ and $g(u) \equiv 1$, then it reduces to a special case of Theorem 4.1 of Kulenovic and Budincevic [5],
- c) if $p_n = 0$, $g(u) \equiv 1$ and $f(u) = u^\alpha$, $0 < \alpha < 1$ then it reduces to Theorem 4.3 of Hooker and Patula [4].

Consider the difference equation

$$\Delta^2 y_n + \frac{1}{n} \Delta y_n + \frac{3(2)^{2n-1/3}(3n-1)}{n(1+3^{2/3}(2)^{2n/3})} f(y_{n+1})g(\Delta y_n) = 0 \quad (n \in Z) \quad (E_1)$$

where $f(u) = u^{1/3}$ and $g(u) = 1 + u^{2/3}$, then all conditions of Theorem 1 are satisfied. Hence, every solution of (E_1) is oscillatory; one such solution is $y_n = (-1)^n 2^n$.

Corollary 2. *Let the conditions (5) and (6) in Theorem 1 be replaced by the condition (c_6) with $\alpha \in (0, 1)$. Then every solution of equation (1) is oscillatory.*

Proof. The proof is similar to that of Theorem 1 and hence is omitted.

Theorem 3. *In addition to conditions $(c_1) - (c_3)$, (c_5) , (2), (3) and (5), suppose that*

$$\int_c^{\pm\infty} \frac{dx}{f(x)g(x)} < \infty \text{ for all } c > 0 \quad (11)$$

and

$$\sum_{n=1}^{\infty} n q_n g\left(\frac{2}{n}\right) = \infty. \quad (12)$$

Then all solutions of equation (1) are oscillatory.

Proof. Suppose that $\{y_n\}$ is a non-oscillatory solution of (1). As before, we may assume that there is an integer $N \geq 0$ such that

$$y_n > 0 \text{ and } \Delta y_n > 0 \text{ for all } n \geq N.$$

As in the proof of Theorem 1, equation (8) is then valid. Since g is non-increasing, we have from Lemma 1

$$g(\Delta y_n) \geq g\left(\frac{2}{n}y_{n+1}\right) \text{ for all } n \geq 2N. \quad (13)$$

Using (13) in (8) and then applying condition (c_5) we obtain

$$\Delta^2 y_n + K_2 q_n f(y_{n+1}) g\left(\frac{2}{n}\right) g(y_{n+1}) \leq 0, \quad n \geq 2N. \quad (14)$$

Multiplying (14) by $(n+1)$ and then summing from $2N$ to $n-1$ we have

$$\sum_{s=2N}^{n-1} \frac{(s+1)\Delta^2 y_s}{f(y_{s+1})g(y_{s+1})} + \sum_{s=2N}^{n-1} K_2 (s+1) q_s g\left(\frac{2}{s}\right) \leq 0. \quad (15)$$

Using the summation by parts formula for the first term in (15), we obtain

$$\begin{aligned} & \frac{n\Delta y_n}{f(y_n)g(y_n)} - \frac{2N\Delta y_{2N}}{f(y_{2N})g(y_{2N})} - \sum_{s=2N}^{n-1} \frac{\Delta y_s}{f(y_{s+1})g(y_{s+1})} \\ & + \sum_{s=2N}^{n-1} \frac{s\Delta(f(y_s)g(y_s))}{f(y_s)f(y_{s+1})g(y_s)g(y_{s+1})} + \sum_{s=2N}^{n-1} K_2 q_s (s+1) g\left(\frac{2}{s}\right) \leq 0 \end{aligned}$$

or

$$\frac{n\Delta y_n}{f(y_n)g(y_n)} - \int_{y_{2N}}^{y_n} \frac{dx}{f(x)g(x)} + K_2 \sum_{s=2N}^{n-1} (s+1) q_s g\left(\frac{2}{s}\right) \leq c \quad (16)$$

where c is a positive constant. Letting $n \rightarrow \infty$ in (16) we obtain a contradiction. This completes the proof.

Remark. In Theorem 3

- (a) if $p_n = 0$, then it reduces to Theorem 2 of [10]
- (b) if $p_n = 0$ and $g(u) \equiv 1$ then it reduces a special case of Theorem 4.2 of Kulenovic and Budincevic [5]
- (c) if $p_n = 0$, $g(u) \equiv 1$ and $f(u) = u^\alpha$, $1 < \alpha < \infty$ then it reduces to Theorem 4.1 of Hooker and Patula [4].

The difference equation

$$\Delta^2 y_n + \frac{1}{n+2} \Delta y_n + \frac{(4n^2 + 6n + 1)(n+1)^2}{n(n+2)} y_{n+1}^3 = 0, \quad n \geq 1 \quad (E_2)$$

satisfies all conditions of Theorem 3 and hence every solution of (E_2) is oscillatory. One such solution is $y_n = \frac{(-1)^n}{n}$.

Corollary 4. *Let conditions (5) and (11) be replaced by condition (c_6) with $\alpha \in [1, \infty)$. Then every solution of equation (1) is oscillatory.*

Proof. The proof is similar to that of Theorem 3 and hence is omitted.

Theorem 5. *Let conditions $(c_1) - (c_3)$, (c_5) , (2) and (5) hold and let in addition condition (c_6) hold for $\alpha = 1$. If there exists a positive sequence $\{h_n\}_{n \geq N}$ such that*

$$\sum_{n=2N}^{\infty} h_n \left[q_n g\left(\frac{2}{n}\right) - \left(\frac{\Delta h_n}{2h_n}\right)^2 \right] = \infty \quad (17)$$

then all solutions of equation (1) are oscillatory.

Proof. Suppose that $\{y_n\}$ is a non-oscillatory solution of (1). In view of Lemma 2 there is no loss in generality in assuming that there is an integer $N \geq 0$ such that

$$y_n > 0 \text{ and } \Delta y_n > 0 \text{ for all } n \geq N.$$

From the proof of Theorem 1, we have the inequality (8) satisfied for all $n \geq N$. Let us define $Z_n = h_n \Delta y_n / y_n$, then from (8), we get ($n \geq N$)

$$\Delta Z_n \leq \frac{-h_n q_n f(y_{n+1}) g(\Delta y_n)}{y_{n+1}} + \frac{\Delta h_n \Delta y_{n+1}}{y_{n+1}} - \frac{h_n \Delta y_n \Delta y_{n+1}}{y_n y_{n+1}}. \quad (18)$$

Since g is non-increasing, we have from Lemma 1 and condition (c_5)

$$g(\Delta y_n) \geq K_2 g\left(\frac{2}{n}\right) g(y_{n+1}), \text{ for all } n \geq 2N. \quad (19)$$

In view of monotonicity of y_n and Δy_n , (19) and (c_6) we see that for $n \geq 2N$

$$\begin{aligned} \Delta Z_n &\leq -\beta K_2 h_n q_n g\left(\frac{2}{n}\right) + Z_{n+1} \frac{\Delta h_n}{h_{n+1}} - Z_{n+1}^2 \frac{h_n}{h_{n+1}^2} \\ &= -\beta K_2 h_n q_n g\left(\frac{2}{n}\right) - \frac{h_n}{h_{n+1}^2} \left[Z_{n+1} - \frac{\Delta h_n h_{n+1}}{2h_n} \right]^2 + \frac{(\Delta h_n)^2}{4h_n}. \end{aligned}$$

Hence

$$\Delta Z_n \leq -\beta K_2 h_n q_n g\left(\frac{2}{n}\right) + \frac{(\Delta h_n)^2}{4h_n} \quad (n \geq 2N).$$

Summing the above inequality from $2N$ to n , we get

$$Z_{n+1} \leq Z_{2N} - \beta K_2 \sum_{s=2N}^n h_s \left[q_s g\left(\frac{2}{s}\right) - \left(\frac{\Delta h_s}{2h_s}\right)^2 \right].$$

Now by (17), it is easy to see that Z_n is essentially negative which is a contradiction.

Remark. In Theorem 5,

- (a) if $p_n = 0$, then it reduces to Theorem 4 of [10]
 (b) if $p_n \equiv 0$ and $g(u) \equiv 1$, then it reduces a result in Szmanda [8].

Consider the difference equation

$$\Delta^2 y_n + \frac{1}{(n+2)} \Delta y_n + \frac{(4n^2 + 6n + 1)}{n^{1/3}(2n+1)^{2/3}} f(y_{n+1})g(\Delta y_n) = 0 \quad (E_3)$$

where $f(u) = u^{1/3}$ and $g(u) = (u)^{2/3}$, then all conditions of Theorem 5 are satisfied. Hence every solution of (E_3) is oscillatory. One such solution is $y_n = \frac{(-1)^n}{n}$.

Finally we establish sufficient conditions for all solutions of (1) to be oscillatory when $\phi(x, y) \equiv y$ and $\{p_n\}$ may be negative.

Theorem 6. Let conditions (c_1) , (c_3) , (c_5) , (2) and (3) hold. Suppose

$$\int^{\pm\infty} \frac{du}{f(u)g(u)} < \infty$$

$$\begin{aligned} f(u) - f(v) &= h_1(u, v)(u - v) : h_1(u, v) > \sigma_1 > 0 \\ g(u) - g(v) &= h_2(u, v)(u - v) : h_2(u, v) > \sigma_2 > 0 \end{aligned} \quad (20)$$

for $u, v \neq 0$ and there is a constant $K > 0$ and a positive non-decreasing sequence $\{h_n\}$ such that

$$\Delta^2 h_n \leq 0 \text{ and } p_n \geq \frac{-K}{h_n} \text{ for all } n \geq N \geq 0. \quad (21)$$

If

$$\sum_{n=2N}^{\infty} h_n q_n g\left(\frac{2}{n}\right) = \infty \quad (22)$$

then all solutions of equation (1) are oscillatory.

Proof. Let $\{y_n\}$ be a non-oscillatory solution of (1) such that $y_n > 0$ and $\Delta y_n > 0$ for $n \geq N \geq 0$. From Lemma 1, we have

$$g(\Delta y_n) \geq K_2 g\left(\frac{2}{n}\right) g(y_{n+1}) \text{ for all } n \geq 2N.$$

Using this inequality, equation (1) becomes

$$\Delta^2 y_n + p_n \Delta y_n + K_2 q_n g\left(\frac{2}{n}\right) f(y_{n+1})g(y_{n+1}) \leq 0 \text{ for } n \geq 2N. \quad (23)$$

Multiplying (23) by $\frac{h_n}{f(y_{n+1})g(y_{n+1})}$ and summing from $2N$ to $n-1$, we have

$$\sum_{s=2N}^{n-1} \frac{h_s \Delta^2 y_s}{f(y_{s+1})g(y_{s+1})} + \sum_{s=2N}^{n-1} \frac{p_s h_s \Delta y_s}{f(y_{s+1})g(y_{s+1})} + K_2 \sum_{s=2N}^{n-1} q_s h_s g\left(\frac{2}{s}\right) \leq 0.$$

Using summation by parts formula and (20) and (21), we have

$$\begin{aligned} & \frac{h_n \Delta y_n}{f(y_{n+1})g(y_{n+1})} - \frac{h_{2N} \Delta y_{2N}}{f(y_{2N+1})g(y_{2N+1})} - \sum_{s=2N}^{n-1} y_{s+1} \Delta \left[\frac{h_s}{f(y_{s+1})g(y_{s+1})} \right] \\ & + \sum_{s=2N}^{n-1} \frac{p_s h_s \Delta y_s}{f(y_{s+1})g(y_{s+1})} + K_2 \sum_{s=2N}^{n-1} q_s h_s g\left(\frac{2}{s}\right) \leq 0 \end{aligned}$$

or

$$\begin{aligned} & \frac{h_n \Delta y_n}{f(y_{n+1})g(y_{n+1})} - \frac{h_{2N} \Delta y_{2N}}{f(y_{2N+1})g(y_{2N+1})} + \sum_{s=2N}^{n-1} \frac{\sigma_1 f(y_{s+2}) + \sigma_2 g(y_{s+1})}{f(y_{s+1})f(y_{s+2})g(y_{s+1})g(y_{s+2})} \\ & - K \int_{y_{2N+1}}^{y_n} \frac{du}{f(u)g(u)} + K_2 \sum_{s=2N}^{n-1} q_s h_s g\left(\frac{2}{s}\right) \leq 0 \end{aligned}$$

or

$$\frac{h_n \Delta y_n}{f(y_{n+1})g(y_{n+1})} \leq c + K \int_{y_{2N+1}}^{y_n} \frac{du}{f(u)g(u)} - K_2 \sum_{s=2N}^{n-1} q_s h_s g\left(\frac{2}{s}\right)$$

where c is a constant. Taking the limit as $n \rightarrow \infty$ and using (22), we arrive at the contradiction that $\Delta y_n < 0$ for all $n \geq 2N$. This completes the proof.

Remark. Let $h_n = n$, and $g(u) \equiv 1$, then Theorem 6 is a discrete analogue of Theorem 4 of Naito [6].

The equation

$$\Delta^2 y_n - \frac{1}{n} \Delta y_n + 12(4)^n \left(\frac{3}{2} + \frac{1}{n}\right) y_{n+1}^3 = 0 \tag{E4}$$

has the oscillatory solution $y_n = (-1)^n n$, since all conditions of Theorem 6 are satisfied.

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