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ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

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Abstract. Let $M_{n,p}(\alpha)$ denote the class of functions of the form

$$f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\cdots+a_{k+p+1}z^k+\cdots\quad (p\in N=\{1,2,3,\ldots\})$$

that are regular in the punctured disk $E=\{z:0<|z|<1\}$ and satisfy

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\}<-\alpha$$

for $0\le\alpha<p$ and $z\in U=\{z:|z|<1\}$, where

$$D^{n+p-1}f(z)=\frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)}$$

N.E. Cho and S. Owa [1] showed that $M_{n+1,p}(\alpha)\subset M_{n,p}(\alpha)$. In this paper, we use the Miller and Mocanu's lemma [3] to improve this property.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\cdots+a_{k+p+1}z^k+\cdots,\ p\in \mathbb{N},$$

which are regular in the punctured disk $E=\{z:0<|z|<1\}$. If $f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k-1}z^{k-p}$ and $g(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}b_{k-1}z^{k-p}$ belong to $\sum_p$, we define the Hadamard

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product of f and g by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k-1}b_{k-1}z^{k-p},\ z\in E.$$

Let

$$D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z),\ z\in E,$$

or, equivalently,

$$\begin{aligned}D^{n+p-1}f(z)=&\frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)}\\ =&\frac{1}{z^p}+(n+p)\frac{a_0}{z^{p-1}}+\frac{(n+p+1)(n+p)}{2!}\frac{a_1}{z^{p-2}}+\cdots\\ &+\frac{(n+k+2p-1)\cdots(n+p)}{(k+p)!}a_{k+p-1}z^k+\cdots\ (z\in E),\end{aligned}$$

where n is any integer greater than $-p$. Let $M_{n,p}(\alpha)$ denote the class of function $f(z)\in\sum_p$ and satisfies

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\}<-\alpha$$

for $0\le\alpha<\ p$ and $z\in U$. In particular, $M_{-p+1,p}(\alpha)$ is the class of meromorphically p-valent starlike functions of order $\alpha(0\le\alpha<\ p)$. N.E. Cho and S. Owa in [1] used Jack's lemma [2] to obtain the following result:

Theorem A. *$M_{n+1,p}(\alpha)\subset M_{n,p}(\alpha)$ for each integer n greater than $-p$.*

In this paper, we use Miller and Mocanu's lemma [3] to improve this result. To prove our results, we need the following lemma [3].

Lemma A. *Let the function $\psi(u,v)$ be a complex valued function, $\psi:D\to\mathbb{C},D\subset\mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane), and let $u=u_1+iu_2$ and $v=v_1+iv_2$. Suppose that the function $\psi(u,v)$ satisfies the following conditions:*
(1) $\psi(u,v)$ is continuous in D;
(2) $(1,0)\in D$ and $\mathrm{Re}\{\psi(1,0)\}>0$;
(3) $\mathrm{Re}\{\psi(iu_2,v_1)\}\le 0$ for all $(iu_2,v_1)\in D$ and such that $u_1\le-\frac{1}{2}(1+u_2^2)$. Let

$$p(z)=1+p_1z+p_2z^2+\cdots$$

be regular in the unit disk U such that $(p(z),zp'(z))\in D$ for all $z\in U$. If

$$\mathrm{Re}\{\psi(p(z),zp'(z))\}>0\ (z\in U),$$

then

$$\mathrm{Re}\{p(z)\} > 0 \ (z \in U).$$

2. Main Results

Theorem 1. *$M_{n+1,p}(\alpha) \subset M_{n,p}\ (\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})})$ for each integer n greater than $-p$.*

Proof. If $f(z) \in M_{n+1,p}(\alpha)$, then

$$\mathrm{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}\right\} < -\alpha, \text{ for } z \in U,$$

equivalently,

$$\mathrm{Re}\left\{-\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}\right\} > \alpha, \text{ for } z \in U. \tag{1}$$

We have to show that (1) implies the inequality

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -(\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})}), \text{ for } z \in U. \tag{2}$$

Let

$$-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = (p-\beta)p(z) + \beta, \text{ where } \beta = \alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})}. \tag{3}$$

Then the function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in U and $p(0) = 1$.

Using the identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z),$$

the form (3) may be written as

$$(n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = -[(p-\beta)p(z) + \beta - n - 2p]. \tag{4}$$

Logarithmic differentiation of (4) yields

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \frac{(p-\beta)zp'(z)}{(p-\beta)p(z) + \beta - n - 2p}. \tag{5}$$

From (3), (5) can be written as

$$-\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = (p-\beta)p(z) + \beta - \frac{(p-\beta)zp'(z)}{(p-\beta)p(z) + \beta - n - 2p}.$$

Let

$$\psi(r,s) = (p-\beta)r + \beta - \alpha - \frac{(p-\beta)s}{(p-\beta)r+\beta-n-2p}.$$

By the hypothesis of (1), we have $\mathrm{Re}\{\psi(p(z), zp'(z))\} > 0$, therefore in order to show that $\mathrm{Re}\{p(z)\} > 0$ for $z \in U$, we have to claim that the function $\psi(r,s)$ satisfies all conditions of Lemma A:

(1) $\psi(r,s)$ is continuous in $D = (\mathbb{C} - \left\{\frac{n+2p-\beta}{(p-\beta)}\right\}) \times \mathbb{C}$;

(2) $(1,0) \in D$ and $\mathrm{Re}\{\psi(1,0)\} = (p-\alpha) > 0$;

(3) for all $(ir,s) \in D$ such that $s \le -\frac{1}{2}(1+r^2)$,

$$\begin{aligned}
\mathrm{Re}\{\psi(ir,s)\} =& \beta - \alpha - \frac{(p-\beta)(\beta-n-2p)s}{(\beta-n-2p)^2+(p-\beta)^2r^2} \\
=& \beta - \alpha + \frac{(p-\beta)(n+2p-\beta)s}{(n+2p-\beta)^2+(p-\beta)^2r^2} \\
\le& \beta - \alpha + \frac{(p-\beta)(n+2p-\beta)}{(n+2p-\beta)^2+(p-\beta)^2r^2} \cdot \frac{-(1+r^2)}{2} \\
=& \frac{(p-\alpha)}{2(n+2p+\frac{1}{2})} - \frac{(p-\alpha)(\frac{n+2p}{n+2p+\frac{1}{2}})(n+2p-\beta)(1+r^2)}{2[(n+2p-\beta)^2+(p-\beta)^2r^2]} \\
=& \frac{-B}{A}\{\beta(n+2p-\beta) + r^2[(p-\beta)(n+p+\beta) + (n+2p)(n+p)]\},
\end{aligned}$$

where $A = 2(n+2p+\frac{1}{2})[(n+2p-\beta)^2+(p-\beta)^2r^2] > 0$, $B = (p-\alpha) > 0$. Since $n > -p$, $0 < \beta < p$ and p is a positive integer, we have $\mathrm{Re}\{\psi(ir,s)\} < 0$. Therefore we have $\mathrm{Re}\{p(z)\} > 0$ for $z \in U$, and hence form (3), we obtain

$$\mathrm{Re}\left\{-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} > \beta = (\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})}),\ z \in U,$$

which equivalently,

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -(\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})}),\ z \in U.$$

Hence $f(z) \in M_{n,p}(\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})})$. We complete this proof.

It is obviously that $M_{n,p}(\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})}) \subset M_{n,p}(\alpha)$, since $0 \le \alpha < p$. Therefore, by Theorem 1, we have $M_{n+1,p}(\alpha) \subset M_{n,p}(\alpha)$, which is the Theorem A in [1], this means that our result improve Theorem A.

Theorem 2. *If $f(z) \in M_{n,p}(\alpha)$. Set*

$$F_c(z) = \frac{1}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \ \textit{with}\ c > 0. \tag{6}$$

Then $F_c(z) \in M_{n,p}(\alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})})$.

Proof. From the hypothesis, we have

$$z(D^{n+p-1}F_c(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F_c(z). \tag{7}$$

Let $f(z) \in M_{n,p}(\alpha)$. We claim that

$$\text{Re}\left\{\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)}\right\} < -(\alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})}).$$

Let

$$-\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = (p-\delta)p(z) + \delta,\ \delta = \alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})}. \tag{8}$$

Then $p(z) = 1 + p_1^* z + p_2^* z^2 + \cdots$ is regular in U and $p(0) = 1$. Using (7), (6) may be written as

$$c\frac{D^{n+p-1}f(z)}{D^{n+p-1}F_c(z)} = -[(p-\delta)p(z) + \delta - c - p]. \tag{9}$$

Differentiating (9) logrithmically and using (8), we obtain

$$-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = (p-\delta)p(z) + \delta - \frac{(p-\delta)zp'(z)}{(p-\delta)p(z) + \delta - c - p}.$$

Define

$$\psi(r,s) = (p-\delta)r + \delta - \alpha - \frac{(p-\delta)s}{(p-\delta)r + \delta - c - p}.$$

By the hypothesis, $\text{Re}\{\psi(p(z), zp'(z))\} > 0$. With the same method as in the proof of Theorem 1, we can show that $\text{Re}\{p(z)\} > 0,\ z \in U$. From (8), we have

$$\text{Re}\left\{-\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)}\right\} < \delta = (\alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})}),\ z \in U,$$

which equivalently,

$$\text{Re}\left\{\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)}\right\} > -(\alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})}),\ z \in U,$$

that is, $F_c(z) \in M_{n,p}(\alpha + \frac{p-\alpha}{2(c+p+\frac{1}{2})})$.

From the Theorem 2, we can get the following corollaries.

Corollary 1. *If* $f(z) \in M_{n,p}(\alpha)$ *then* $F_c(z) \in M_{n,p}(\alpha)$, *where* $F(z)$ *is defined by (6).*

Corollary 2. *If* $f(z) \in M_{n,p}(\alpha)$ *then* $F_{n+p}(z) \in M_{n,p}(\alpha + \frac{p-\alpha}{2(n+2p+\frac{1}{2})})$.

Corollary 3. *If* $f(z) \in M_{n,p}(\alpha)$ *then* $F_{n+p}(z) \in M_{n,p}(\alpha)$.

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