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A NOTE ON CLASSICAL HILBERT INEQUALITY

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Abstract. In this note we establish some new generalizations of the classical inequality of Hilbert with non-conjugate parameters.

1. Introduction

Let $x_m \geq 0$, $x_m \not\equiv 0$, $y_n \geq 0$, $y_n \not\equiv 0$, $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$
The inequality:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m y_n}{m+n} < \pi \csc\left(\frac{\pi}{p}\right) \left(\sum_{m=1}^{\infty} x_m{}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} y_n{}^{p'}\right)^{\frac{1}{p'}} \tag{1}$$

is well-known generalization of the classical inequality of Hilbert, where the constant $\pi \csc(\frac{\pi}{p})$ is best possible. (see for instance Th.315 from [2]).

The case when instead of p, p' we have non-conjugate parameters p, q was also consider in [2].

If

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} \geq 1,$$

such that

$$0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} = \frac{1}{p'} + \frac{1}{q'} \leq 1,$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m y_n}{(m+n)^{\lambda}} < K(p,q,\lambda) \left(\sum_{m=1}^{\infty} x_m{}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} y_n{}^q\right)^{\frac{1}{q}},$$

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where

$$K(p,q,\lambda) = \left(\pi \csc \frac{\pi}{\lambda p'}\right)^{\lambda} = \left(\pi \csc \frac{\pi}{\lambda q'}\right)^{\lambda} \text{[see1, 4, 5]}.$$

In [3] Mitrinović and Pečarić proved that, if

$$\lambda > 0, \quad \frac{1}{p} + \frac{1}{q} > \lambda \geq \frac{1}{p'} + \frac{1}{q'}$$

then

$$K(p,q,\lambda) = \left(\pi \csc \left(\frac{\pi}{2\lambda}\left(\frac{1}{p} - \frac{1}{q} + \lambda\right)\right)\right)^{\lambda}.$$

which follows from (1) and the following

Theorem A. *Let $a_{mn} \geq 0$ and suppose that for every $p > 1$ and $x_m \geq 0, y_n \geq 0$ the follwoing inequality is valid*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}x_m y_n \leq K(p)\left(\sum_{m=1}^{\infty} x_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} y_n^{p'}\right)^{\frac{1}{p'}}$$

where p', p are conjugate i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

If $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} \geq 1$ and $\lambda > 0, \frac{1}{p} + \frac{1}{q} > \lambda \geq \frac{1}{p'} + \frac{1}{q'}$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{\lambda}x_m y_n \leq K^{\lambda}\left(\left(\frac{1}{2\lambda}\left(\frac{1}{p} - \frac{1}{q} + \lambda\right)\right)^{-1}\right)\left(\sum_{m=1}^{\infty} x_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} y_n^q\right)^{\frac{1}{q}}$$

The following theorems are stated without proof in [3].

Theorem B. *Let $k(x,y) \geq 0$ and suppose that for every $p > 1$ and $f(x) \geq 0, g(y) \geq 0$ the following inequality is valid*

$$\int\int k(x,y)f(x)g(y)dxdy \leq K(p)\left(\int f^p(x)dx\right)^{\frac{1}{p}}\left(\int g^{p'}(y)dy\right)^{\frac{1}{p'}}.$$

If $p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} \geq 1$ and $\lambda > 0, \quad \lambda = \frac{1}{p'} + \frac{1}{q'}$, then

$$\int\int k^{\lambda}(x,y)f(x)g(y)dxdy \leq K^{\lambda}(\lambda q')\left(\int f^p(x)dx\right)^{\frac{1}{p}}\left(\int g^q(y)dy\right)^{\frac{1}{q}}.$$

Theorem C. *Let $a_{mn\ldots s} \geq 0$ and suppose that for every $p_1 > 1,\ p_2 > 1, \cdots, p_r > 1$ s.t. $\sum_{i=1}^{r} p_i^{-1} = 1$ and $x_m \geq 0,\ y_n \geq 0, \ldots,\ z_s \geq 0$ the following inequality is* *valid*

$$\sum a_{mn\ldots s}x_m y_n \ldots z_s \le K(p_1,p_2,\ldots,p_r)\left(\sum_{m=1}^{\infty} x_m{}^{p_1}\right)^{\frac{1}{p_1}}\left(\sum_{n=1}^{\infty} y_n{}^{p_2}\right)^{\frac{1}{p_2}} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{p_r}\right)^{\frac{1}{p_r}}.$$

If $q_i > 1$, $i = 1,2,\ldots,r$ and $\lambda > 0$, $(r-1)\lambda \ge \sum_{i=1}^{r} 1/q_i'$, $\sum_{j=1}^{r} 1/q_j > \lambda$, $i = 1,2,\ldots,r$, then

$$\sum a^{\lambda}_{mn\ldots s}x_m y_n \cdots z_s \le K^{\lambda}\left(\underline{p}_1,\underline{p}_2,\ldots,\underline{p}_r\right)\left(\sum_{m=1}^{\infty} x_m{}^{q_1}\right)^{\frac{1}{q_1}}\left(\sum_{n=1}^{\infty} y_n{}^{q_2}\right)^{\frac{1}{q_2}} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{q_r}\right)^{\frac{1}{q_r}},$$

where $\frac{1}{\underline{p}_i} = \frac{1}{r\lambda}\left(\frac{r}{q_i} - \sum_{j=1}^{r}\frac{1}{q_j} + \lambda\right), i = 1,2,\ldots,r.$

Theorem D. *Let $k(x,y,\ldots,z) \ge 0$ and suppose that for every $p_1 > 1$, $p_2 > 1, \cdots, p_r > 1$ s.t. $\sum_{i=1}^{r} p_i^{-1} = 1$ and $f(x) \ge 0$, $g(y) \ge 0, \ldots, h(z) \ge 0$ the following inequality is valid*

$$\int\int\cdots\int k(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz \le K(p_1,p_2,\ldots,p_r)$$

$$\left(\int f^{p_1}(x)dx\right)^{\frac{1}{p_1}}\cdot\left(\int g^{p_2}(y)dy\right)^{\frac{1}{p_2}}\cdots\left(\int h^{p_r}(z)dz\right)^{\frac{1}{p_r}}.$$

If $q_i > 1$, $i = 1,2,\ldots,r$ and $0 < \lambda < 1$, $\sum_{i=1}^{r}\frac{1}{q_i} > 1$, $(r-1)\lambda = \sum_{i=1}^{r}\frac{1}{q_i'}$, then

$$\int\int\cdots\int k^{\lambda}(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz \le K^{\lambda}(\overline{p}_1,\overline{p}_2,\ldots,\overline{p}_r)$$

$$\left(\int f^{q_1}(x)dx\right)^{\frac{1}{q_1}}\left(\int g^{q_2}(y)dy\right)^{\frac{1}{q_2}}\cdots\left(\int h^{q_r}(z)dz\right)^{\frac{1}{q_r}},$$

where $\overline{p}_i = \lambda(\lambda - 1 + \frac{1}{q_i})^{-1}$, $i = 1,2,\ldots,r$.

The aim of this note is to extended Theorem A and give proofs of Theorem B, Theorem C and Theorem D in general forms.

2.Main results

Theorem 1. *Let $c > 0$, $s \ge 0$, $t \ge 0$ s.t. $s + c \ge 1$, $t + c \ge 1$ and $f(m) \ge 0, g(n) \ge 0$*

Let $a_{mn} \geq 0$ and suppose that for every $p > 1$ and $x_m \geq 0$, $y_n \geq 0$ the following inequality is valid

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}x_m y_n \leq K\left(\frac{p}{c}\right)\left(\sum_{m=1}^{\infty} x_m^{\frac{p}{c}}\right)^{\frac{c}{p}}\left(\sum_{n=1}^{\infty} y_n^{\frac{p'}{c}}\right)^{\frac{c}{p'}}.$$

If $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\lambda > 0$, $\frac{1}{p} + \frac{1}{q} > \lambda \geq \frac{1}{p'} + \frac{1}{q'}$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{\lambda}(f^s(m)g^t(n))^{k_2}x_m y_n \leq K^{\lambda}\left(\left(\frac{c}{2\lambda}\left(\frac{1}{p} - \frac{1}{q} + \lambda\right)\right)^{-1}\right)$$

$$\left(\sum_{m=1}^{\infty} f(m)\right)^{sk_2}\left(\sum_{n=1}^{\infty} g(n)\right)^{tk_2}\left(\sum_{m=1}^{\infty} x_m^{\frac{p}{c}}\right)^{\frac{c}{p}}\left(\sum_{n=1}^{\infty} y_n^{\frac{q}{c}}\right)^{\frac{c}{q}},$$

where $k_2 = \frac{1}{2}(\frac{1}{p} + \frac{1}{q} - \lambda)$

Proof. We need the following

Lemma 1. (*see* [3]) *Let* $a_{mn} \geq 0$, $b_{mn} \geq 0$, $u_1 > 0$, $v_1 > 0$, $u_2 > 0$, $v_2 > 0$ *and suppose that for every nonnegative* x_m, y_n *the following inequalities are valid*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}x_m y_n \leq K_1\left(\sum_{m=1}^{\infty} x_m^{\frac{1}{u_1}}\right)^{u_1}\left(\sum_{n=1}^{\infty} y_n^{\frac{1}{v_1}}\right)^{v_1},$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}x_m y_n \leq K_2\left(\sum_{m=1}^{\infty} x_m^{\frac{1}{u_2}}\right)^{u_2}\left(\sum_{n=1}^{\infty} y_n^{\frac{1}{v_2}}\right)^{v_2}.$$

Futher, let $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 \geq 1$, $u = k_1u_1 + k_2u_2$, $v = k_1v_1 + k_2v_2$.
Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{k_1}b_{mn}^{k_2}x_m y_n \leq {K_1}^{k_1}{K_2}^{k_2}\left(\sum_{m=1}^{\infty} x_m^{\frac{1}{u}}\right)^{u}\left(\sum_{n=1}^{\infty} y_n^{\frac{1}{v}}\right)^{v}.$$

Proof of Theorem 1. If we set $b_{mn} = f^s(m)g^t(n), u_1 = \frac{c}{p_1}$, $v_1 = \frac{c}{p_1'}$, $u_2 = v_2 = c$, $k_1 = \lambda$, $u = \frac{c}{p}$, $v = \frac{c}{q}$.

Since

$$\begin{cases} u = k_1u_1 + k_2u_2 \\ v = k_1v_1 + k_2v_2 \end{cases},$$

we have

$$\begin{cases} \frac{1}{p} = \lambda\frac{1}{p_1} + k_2 \\ \frac{1}{q} = \lambda\frac{1}{p_1{}'} + k_2 \end{cases}.$$

So that $k_2 = \frac{1}{2}(\frac{1}{p} + \frac{1}{q} - \lambda)$ and $\frac{1}{p_1} = \frac{1}{2\lambda}(\frac{1}{p} - \frac{1}{q} + \lambda)$.

The conditions of Lemma are fulfield, since we have:
$k_2 > 0$ and $k_1 + k_2 = \frac{1}{2}(\frac{1}{p} + \frac{1}{q} + \lambda) \geq \frac{1}{2}(\frac{1}{p} + \frac{1}{q} + \frac{1}{p'} + \frac{1}{q'}) = 1$.

Now to check $p_1 > 1$ where $p_1 = 2\lambda/\frac{1}{p} - \frac{1}{q} + \lambda$, we observe
$\langle i \rangle$ $\frac{1}{p} - \frac{1}{q} + \lambda \geq \frac{1}{p} - \frac{1}{q} + \frac{1}{p'} + \frac{1}{q'} = 1 - \frac{1}{q} + \frac{1}{q'} = \frac{2}{q'} > 0$
and $\langle ii \rangle$ $2\lambda - (\frac{1}{p} - \frac{1}{q} + \lambda) = \lambda - \frac{1}{p} + \frac{1}{q} \geq \frac{1}{p'} + \frac{1}{q'} - \frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{p} + \frac{1}{p'} = \frac{2}{p'} > 0$.

It follows from $\langle i \rangle$ and $\langle ii \rangle$ that $p_1 > 1$. Now we have:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}x_m y_n \leq & K\left(\frac{p_1}{c}\right)\left(\sum_{m=1}^{\infty} x_m^{\frac{p_1}{c}}\right)^{\frac{c}{p_1}}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{p_1{}'}{c}}\right)^{\frac{c}{p_1{}'}}\\ =& K_1\left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u_1}}\right)^{u_1}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v_1}}\right)^{v_1},\end{aligned}$$

where $K_1 = K(\frac{p_1}{c})$.

Futhermore, Since $s + c \geq 1$, $t + c \geq 1$, we have (see [1], p29)

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f^s(m)g^t(n)x_m y_n\\ =&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f^s(m)g^t(n)\left(x_m{}^{\frac{1}{c}}\right)^c\left(y_n{}^{\frac{1}{c}}\right)^c\\ \leq&\left(\sum_{m=1}^{\infty} f^s(m)\left(x_m{}^{\frac{1}{c}}\right)^c\right)\left(\sum_{n=1}^{\infty} g^t(n)\left(y_n{}^{\frac{1}{c}}\right)^c\right)\\ \leq&\left(\sum_{m=1}^{\infty} f(m)\right)^s\left(\sum_{n=1}^{\infty} x_m{}^{\frac{1}{c}}\right)^c\left(\sum_{n=1}^{\infty} g(n)\right)^t\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{c}}\right)^c\\ =&K_2\left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u_2}}\right)^{u_2}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v_2}}\right)^{v_2},\end{aligned}$$

where $K_2 = (\sum_{m=1}^{\infty} f(m))^s(\sum_{n=1}^{\infty} g(n))^t$

It follows from Lemma 1 that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^{\lambda}(f^s(m)g^t(n))^{k_2}x_m y_n \leq K^{\lambda}\left(\left(\frac{c}{2\lambda}\left(\frac{1}{p} - \frac{1}{q} + \lambda\right)\right)^{-1}\right)$$

$$\left(\sum_{m=1}^{\infty} f(m)\right)^{sk_2}\left(\sum_{n=1}^{\infty} g(n)\right)^{tk_2}\left(\sum_{m=1}^{\infty} x_m^{\frac{p}{c}}\right)^{\frac{c}{p}}\left(\sum_{n=1}^{\infty} y_n^{\frac{q}{c}}\right)^{\frac{c}{q}},$$

whrer $k_2 = \frac{1}{2}(\frac{1}{p} + \frac{1}{q} - \lambda)$. This completes the proof.

Theorem 2. *Let $c \in (0,1]$, $s \geq 0$, $t \geq 0$ s.t. $s + c = 1$, $t + c = 1$ and $w(x), r(y)$ are nonnegative integrable functions.*

Let $k(x,y) \geq 0$ and suppose that for every $p > 1$ and $f(x) \geq 0$, $g(y) \geq 0$ the following inequality is valid

$$\int\int k(x,y)f(x)g(y)dxdy \leq K\left(\frac{p}{c}\right)\left(\int f^{\frac{p}{c}}(x)dx\right)^{\frac{c}{p}}\left(\int g^{\frac{p'}{c}}(y)dy\right)^{\frac{c}{p'}}.$$

If $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\lambda > 0$, $\lambda = \frac{1}{p'} + \frac{1}{q'}$, then

$$\int\int k^{\lambda}(x,y)(w^s(x)r^t(y))^{1-\lambda}f(x)g(y)dxdy \leq K^{\lambda}\left(\frac{\lambda q'}{c}\right)\left(\int w(x)dx\right)^{s(1-\lambda)}$$

$$\left(\int r(y)dy\right)^{t(1-\lambda)}\left(\int f^{\frac{p}{c}}(x)dx\right)^{\frac{c}{p}}\left(\int g^{\frac{q}{c}}(y)dy\right)^{\frac{c}{q}}.$$

Proof. First, we need the following:

Lemma 2. *Let $k(x,y) \geq 0$, $m(x,y) \geq 0$, $u_1 > 0$, $v_1 > 0$, $u_2 > 0$, $v_2 > 0$ and suppose that for every nonnegative $f(x), g(y)$ the following inequalities are valid*

$$\int\int k(x,y)f(x)g(y)dxdy \leq K_1\left(\int f(x)^{\frac{1}{u_1}}dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v_1}}dy\right)^{v_1} \tag{2}$$

$$\int\int m(x,y)f(x)g(y)dxdy \leq K_2\left(\int f(x)^{\frac{1}{u_2}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v_2}}dy\right)^{v_2}. \tag{3}$$

Futher, let $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 = 1$, $u = k_1u_1 + k_2u_2$, $v = k_1v_1 + k_2v_2$. Then

$$\int\int k^{k_1}(x,y)m^{k_2}(x,y)f(x)g(y)dxdy \leq K_1^{k_1}K_2^{k_2}\left(\int f(x)^{\frac{1}{u}}dx\right)^{u}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v}.$$

Proof of Lemma 2. By Hölder inequality, since $k_1 + k_2 = 1$, it follows

$$\begin{aligned}
&\int\int k^{k_1}(x,y)m^{k_2}(x,y)f(x)g(y)dxdy \\
&= \int\int k^{k_1}(x,y)m^{k_2}(x,y)f^{\frac{k_1u_1+k_2u_2}{u}}(x)g^{\frac{k_1v_1+k_2v_2}{v}}(y)dxdy \\
&= \int\int \left(k(x,y)f^{\frac{u_1}{u}}(x)g^{\frac{v_1}{v}}(y)\right)^{k_1}\left(m(x,y)f^{\frac{u_2}{u}}(x)g^{\frac{v_2}{v}}(y)\right)^{k_2}dxdy \\
&\leq \left(\int\int k(x,y)f^{\frac{u_1}{u}}(x)g^{\frac{v_1}{v}}(x)dxdy\right)^{k_1}\left(\int\int m(x,y)f^{\frac{u_2}{u}}(x)g^{\frac{v_2}{v}}(y)dxdy\right)^{k_2}.
\end{aligned} \tag{4}$$

As consequences of (2) and (3) we have

$$\int\int k(x,y)f^{\frac{u_1}{u}}(x)g^{\frac{v_1}{v}}(y)dxdy \le K_1\left(\int f(x)^{\frac{1}{u}}dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v_1}$$

$$\int\int m(x,y)f^{\frac{u_2}{u}}(x)g^{\frac{v_2}{v}}(y)dxdy \le K_2\left(\int f(x)^{\frac{1}{u}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v_2}.$$

Substituting these inequalities into (4) we obtain:

$$\begin{aligned}
&\int\int k^{k_1}(x,y)m^{k_2}(x,y)f(x)g(y)dxdy\\
\le&\left(K_1\left(\int f(x)^{\frac{1}{u}}dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v_1}\right)^{k_1}\\
&\left(K_2\left(\int f(x)^{\frac{1}{u}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v_2}\right)^{k_2}\\
=&K_1^{k_1}K_2^{k_2}\left(\int f(x)^{\frac{1}{u}}dx\right)^{k_1u_1+k_2u_2}\left(\int g(y)^{\frac{1}{v}}dy\right)^{k_1v_1+k_2v_2}\\
=&K_1^{k_1}K_2^{k_2}\left(\int f(x)^{\frac{1}{u}}dx\right)^{u}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v}.
\end{aligned}$$

Proof of Theorem 2. If we set: $m(x,y) = w^s(x)r^t(y), u_1 = \frac{c}{p_1}$, $v_1 = \frac{c}{p_1'}$, $u_2 = v_2 = c$, $k_1 = \lambda$, $u = \frac{c}{p}$, $v = \frac{c}{q}$, Since

$$\begin{cases} u = k_1u_1 + k_2u_2 \\ v = k_1v_1 + k_2v_2 \end{cases},$$

we have

$$\begin{cases} \frac{1}{p} = \lambda\frac{1}{p_1} + k_2 \\ \frac{1}{q} = \lambda\frac{1}{p_1{}'} + k_2 \end{cases}.$$

$$\begin{aligned}
\text{So that } k_2 = 1-\lambda \text{ and } p_1 =&\lambda\left(\frac{1}{p} - k_2\right)^{-1}\\
=&\lambda\left(\frac{1}{p} - 1 + \lambda\right)^{-1}\\
=&\lambda\left(\frac{1}{p} - 1 + \frac{1}{p'} + \frac{1}{q'}\right)^{-1}\\
=&\lambda\left(-\frac{1}{p'} + \frac{1}{p'} + \frac{1}{q'}\right)^{-1}\\
=&\lambda q' = \left(\frac{1}{p'} + \frac{1}{q'}\right)q' = \frac{q'}{p'} + 1 > 1.
\end{aligned}$$

Now we have:

$$\begin{aligned}\int\int k(x,y)f(x)g(y)dxdy \leq& K\left(\frac{p_1}{c}\right)\left(\int f^{\frac{p_1}{c}}(x)dx\right)^{\frac{c}{p_1}}\left(\int g^{\frac{p_1'}{c}}(y)dy\right)^{\frac{c}{p_1'}}\\ =&K_1\left(\int f(x)^{\frac{1}{u_1}}dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v_1}}dy\right)^{v_1},\end{aligned}$$

where $K_1 = K(\frac{p_1}{c})$.

Since $s + c = 1$, $t + c = 1$, it follows from Hölder inequality that:

$$\begin{aligned}&\int\int w^s(x)r^t(y)f(x)g(y)dxdy\\ =&\int\int w^s(x)r^t(y)\left(f^{\frac{1}{c}}(x)\right)^c\left(g^{\frac{1}{c}}(y)\right)^c dxdy\\ \leq&\left(\int w^s(x)\left(f^{\frac{1}{c}}(x)\right)^c dx\right)\left(\int r^t(y)\left(g^{\frac{1}{c}}(y)\right)^c dy\right)\\ \leq&\left(\int w(x)dx\right)^s\left(\int f^{\frac{1}{c}}(x)dx\right)^c\left(\int r(y)dy\right)^t\left(\int g^{\frac{1}{c}}(y)dy\right)^c\\ =&K_2\left(\int f(x)^{\frac{1}{u_2}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v_2}}dy\right)^{v_2},\end{aligned}$$

where $K_2 = (\int w(x)dx)^s(\int r(y)dy)^t$.

Applying Lemma 2 to obtain

$$\begin{aligned}\int\int k^\lambda(x,y)(w^s(x)r^t(y))^{1-\lambda}f(x)g(y)dxdy \leq K^\lambda\left(\frac{\lambda q'}{c}\right)\left(\int w(x)dx\right)^{s(1-\lambda)}\\ \left(\int r(y)dy\right)^{t(1-\lambda)}\left(\int f^{\frac{p}{c}}(x)dx\right)^{\frac{c}{p}}\left(\int g^{\frac{q}{c}}(y)dy\right)^{\frac{c}{q}}.\end{aligned}$$

This completes the proof of Theorem 2.

The followings are the multidimensional generlization of Theorem 1 and Theorem 2.

Note. In theorem C and D implicitly is given that $\underline{p}_i \geq 1$. Hence, we can set, instead of $\sum_{i=1}^r 1/q_i > \lambda$, the condition $\lambda + r/q_i > \sum_{i=1}^r 1/q_i > \lambda, i = 1, 2, \ldots, r$, in Theorem C, and instead of $\sum_{i=1}^r 1/q_i > 1$, the condition

$$1 + \frac{r-1}{q_i} > \sum_{i=1}^r 1/q_i > 1$$

for $i = 1, 2, \ldots, r$ in the Theorem D.

Theorem 3. *Let $c > 0$, $\alpha \geq 0$, $\beta \geq 0, \ldots$, $\gamma \geq 0$ such that $\alpha + c \geq 1$, $\beta + c \geq 1$, $\ldots, \gamma + c \geq 1$ and $f(m)$, $g(n), \ldots$, $h(s)$ are nonnegative.*

Let $a_{mn\cdots s} \geq 0$ and suppose that for every $p_1 > 1, p_2 > 1, \ldots, p_r > 1$ s.t. $\sum_{i=1}^{r} p_i{}^{-1} = 1$ and $x_m \geq 0, y_n \geq 0, \ldots, z_s \geq 0$ the follwoing inequality is valid

$$\sum a_{mn\cdots s} x_m y_n \cdots z_s$$
$$\leq K\left(\frac{p_1}{c}, \frac{p_2}{c}, \ldots, \frac{p_r}{c}\right)\left(\sum_{m=1}^{\infty} x_m{}^{\frac{p_1}{c}}\right)^{\frac{c}{p_1}}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{p_2}{c}}\right)^{\frac{c}{p_2}} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{p_r}{c}}\right)^{\frac{c}{p_r}}$$

If $q_i > 1,\ i = 1, 2, \ldots, r$ and $\lambda > 0, (r-1)\lambda \geq \sum_{i=1}^{r} 1/q_i{}', \lambda + \frac{r}{q_i} > \sum_{j=1}^{r} 1/q_j > \lambda, i = 1, 2, \ldots, r$, then

$$\sum a_{mn\cdots s}^{\lambda} (f^{\alpha}(m) g^{\beta}(n) \cdots h^{\gamma}(s))^{k_2} x_m y_n \cdots z_s \leq K^{\lambda}\left(\frac{\underline{p}_1}{c}, \frac{\underline{p}_2}{c}, \ldots, \frac{\underline{p}_r}{c}\right)$$
$$\left(\sum_{m=1}^{\infty} f(m)\right)^{\alpha k_2}\left(\sum_{n=1}^{\infty} g(n)\right)^{\beta k_2} \cdots \left(\sum_{s=1}^{\infty} h(s)\right)^{\gamma k_2}$$
$$\left(\sum_{m=1}^{\infty} x_m{}^{\frac{q_1}{c}}\right)^{\frac{c}{q_1}}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{q_2}{c}}\right)^{\frac{c}{q_2}}\left(\sum_{s=1}^{\infty} z_a{}^{\frac{q_r}{c}}\right)^{\frac{c}{q_r}}$$

where $\frac{1}{\underline{p}_i} = \frac{1}{r\lambda}\left(\frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda\right), i = 1, 2, \ldots, r, k_2 = \left(\sum_{i=1}^{r} \frac{1}{q_i} - \lambda\right)/r$

Proof. We need the following :

Lemma 3. *Let $a_{mn\cdots s} \geq 0, b_{mn\cdots s} \geq 0, u_1 > 0, v_1 > 0, \ldots, w_1 > 0, u_2 > 0, v_2 > 0, \ldots, w_2 > 0$ and suppose that for every nonnegative $x_m, y_n, \ldots, z_s$ the following inequalities are valid*

$$\sum a_{mn\cdots s} x_m y_n \cdots z_s \leq K_1 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u_1}}\right)^{u_1}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v_1}}\right)^{v_1} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w_1}}\right)^{w_1} \quad (5)$$

$$\sum b_{mn\cdots s} x_m y_n \cdots z_s \leq K_2 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u_2}}\right)^{u_2}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v_2}}\right)^{v_2} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w_2}}\right)^{w_2} \quad (6)$$

Futher, let $k_1 > 0, k_2 > 0, k_1 + k_2 \geq 1, u = k_1 u_1 + k_2 u_2, v = k_1 v_1 + k_2 v_2, \ldots, w = k_1 w_1 + k_2 w_2$. Then

$$\sum a_{mn\cdots s}^{k_1} b_{mn\cdots s}^{k_2} x_m y_n \cdots z_s \leq K_1^{k_1} K_2^{k_2} \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u}\left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w}.$$

Proof of Lemma 3. Using a generalization of Hölder inequality (see [1], p29).

Since $k_1 + k_2 \geq 1$, it follows

$$\begin{aligned}
&\sum a_{mn\cdots s}^{k_1} b_{mn\cdots s}^{k_2} x_m y_n \cdots z_s \\
=&\sum \left(a_{mn\cdots s} x_m{}^{\frac{u_1}{u}} y_m{}^{\frac{v_1}{v}} \cdots z_s{}^{\frac{w_1}{w}}\right)^{k_1} \left(b_{mn\cdots s} x_m{}^{\frac{u_2}{u}} y_n{}^{\frac{v_2}{v}} \cdots z_s{}^{\frac{w_2}{w}}\right)^{k_2} \\
\leq&\left(\sum a_{mn\cdots s} x_m{}^{\frac{u_1}{u}} y_m{}^{\frac{v_1}{v}} \cdots z_s{}^{\frac{w_1}{w}}\right)^{k_1} \left(\sum b_{mn\cdots s} x_m{}^{\frac{u_2}{u}} y_n{}^{\frac{v_2}{v}} \cdots z_s{}^{\frac{w_2}{w}}\right)^{k_2}. \qquad (7)
\end{aligned}$$

As consequences of (5) and (6) we have:

$$\begin{aligned}
&\sum a_{mn\cdots s} x_m{}^{\frac{u_1}{u}} y_m{}^{\frac{v_1}{v}} \cdots z_s{}^{\frac{w_1}{w}} \\
&\leq K_1 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u_1} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v_1} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w_1} \\
&\sum b_{mn\cdots s} x_m{}^{\frac{u_2}{u}} y_n{}^{\frac{v_2}{v}} \cdots z_s{}^{\frac{w_2}{w}} \\
&\leq K_2 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u_2} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v_2} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w_2}.
\end{aligned}$$

Substituting these inequalities into (7) we obtain:

$$\begin{aligned}
&\sum a_{mn\cdots s}^{k_1} b_{mn\cdots s}^{k_2} x_m y_n \cdots z_s \\
\leq&\left(K_1 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u_1} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v_1} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w_1}\right)^{k_1} \\
&\left(K_2 \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u_2} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v_2} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w_2}\right)^{k_2} \\
=&K_1^{k_1} K_2^{k_2} \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{k_1 u_1 + k_2 u_2} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{k_1 v_1 + k_2 v_2} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{k_1 w_1 + k_2 w_2} \\
=&K_1^{k_1} K_2^{k_2} \left(\sum_{m=1}^{\infty} x_m{}^{\frac{1}{u}}\right)^{u} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v}}\right)^{v} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w}}\right)^{w}.
\end{aligned}$$

Proof of Theorem 3. If we set $b_{mn\cdots s} = f^{\alpha}(m) g^{\beta}(n) \cdots h^{\gamma}(s)$, $u_1 = \frac{c}{p_1}$, $v_1 = \frac{c}{p_2}, \ldots, w_1 = \frac{c}{p_r}$ s.t. $\sum_{i=1}^{r} \frac{1}{p_i} = 1$, $u_2 = v_2 = \cdots = w_2 = c$, $k_1 = \lambda$, $u = \frac{c}{q_1}$, $v = \frac{c}{q_2}, \ldots, w = \frac{c}{q_r}$. Since

$$\begin{cases}
u = k_1 u_1 + k_2 u_2 \\
v = k_1 v_1 + k_2 v_2 \\
\vdots \\
w = k_1 w_1 + k_2 w_2,
\end{cases}$$

we have

$$\begin{cases} \frac{1}{q_1} = \lambda \frac{1}{\underline{p}_1} + k_2 \\ \frac{1}{q_2} = \lambda \frac{1}{\underline{p}_1} + k_2 \\ \vdots \\ \frac{1}{q_r} = \lambda \frac{1}{\underline{p}_r} + k_2. \end{cases}$$

So that $k_2 = \frac{\sum_{i=1}^{r} \frac{1}{q_i} - \lambda}{r} \geq 0$ *and*

$$\frac{1}{\underline{p}_i} = \frac{1}{\lambda}\left(\frac{1}{q_i} - \frac{\sum_{j=1}^{r} \frac{1}{q_j} - \lambda}{r}\right)$$

$$= \frac{1}{r\lambda}\left(\frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda\right), \quad i = 1, 2, \ldots, r.$$

The conditions of Lemma 3 are fulfield, since we have:

$$k_2 \geq 0 \text{ and } k_1 + k_2 = \lambda + \frac{\sum_{i=1}^{r} \frac{1}{q_i} - \lambda}{r} = \frac{\sum_{i=1}^{r} \frac{1}{q_i} + \lambda(r-1)}{r}$$

$$\geq \frac{\sum_{i=1}^{r} \frac{1}{q_i} + \sum_{i=1}^{r} \frac{1}{q_i'}}{r} = \frac{r}{r} = 1.$$

Now to check $\underline{p}_i > 1$, where $\underline{p}_i = r\lambda\left(\frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda\right)^{-1}, i = 1, 2, \ldots, r.$

We observe $\langle i \rangle \frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda > 0$

and $\langle ii \rangle r\lambda - \left(\frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda\right) = \sum_{j=1}^{r} \frac{1}{q_j} - \frac{r}{q_i} + \lambda(r-1)$

$$\geq \sum_{j=1}^{r} \frac{1}{q_j} - \frac{r}{q_i} + \sum_{j=1}^{r} \frac{1}{q_j'}$$

$$= r - \frac{r}{q_i} = r\left(1 - \frac{1}{q_i}\right) > 0 \qquad i = 1, 2, \ldots, r.$$

It follows from $\langle i \rangle$ and $\langle ii \rangle$, that $\underline{p}_i > 1, i = 1, 2, \ldots, r$. Now we have

$$\sum a_{mn\cdots s} x_m y_n \cdots z_s \leq K\left(\frac{\underline{p}_1}{c}, \frac{\underline{p}_2}{c}, \ldots, \frac{\underline{p}_r}{c}\right)\left(\sum_{m=1}^{\infty} x_m^{\frac{\underline{p}_1}{c}}\right)^{\frac{c}{\underline{p}_1}}\left(\sum_{n=1}^{\infty} y_n^{\frac{\underline{p}_2}{c}}\right)^{\frac{c}{\underline{p}_2}}$$

$$\cdots \left(\sum_{s=1}^{\infty} z_s^{\frac{\underline{p}_r}{c}}\right)^{\frac{c}{\underline{p}_r}}$$

$$= K_1\left(\sum_{m=1}^{\infty} x_m^{\frac{1}{u_1}}\right)^{u_1}\left(\sum_{n=1}^{\infty} y_n^{\frac{1}{v_1}}\right)^{v_1} \cdots \left(\sum_{s=1}^{\infty} z_s^{\frac{1}{w_1}}\right)^{w_1},$$

where $K_1 = K\left(\frac{p_1}{c}, \frac{p_2}{c}, \ldots, \frac{p_r}{c}\right)$.

Futhermore, Since $\alpha + c \geq 1, \beta + c \geq 1, \ldots, \gamma + c \geq 1$, we have (see [1], p29)

$$\begin{aligned}
&\sum f^{\alpha}(m) g^{\beta}(n) \cdots h^{\gamma}(s) x_m y_n \cdots z_s \\
=&\sum f^{\alpha}(m) g^{\beta}(n) \cdots h^{\gamma}(s) \left(x_m{}^{\frac{1}{c}}\right)^c \left(y_n{}^{\frac{1}{c}}\right)^c \cdots \left(z_s{}^{\frac{1}{c}}\right)^c \\
\leq&\left(\sum f^{\alpha}(m) \left(x_m{}^{\frac{1}{c}}\right)^c\right) \left(\sum g^{\beta}(n) \left(y_n{}^{\frac{1}{c}}\right)^c\right) \cdots \left(\sum h^{\gamma}(s) \left(z_s{}^{\frac{1}{c}}\right)^c\right) \\
\leq&\left(\sum f(m)\right)^{\alpha} \left(\sum x_m{}^{\frac{1}{c}}\right)^c \left(\sum g(n)\right)^{\beta} \left(\sum y_n{}^{\frac{1}{c}}\right)^c \cdots \left(\sum h(s)\right)^{\gamma} \left(\sum z_s{}^{\frac{1}{c}}\right)^c \\
=&K_2 \left(\sum_{m=1}^{\infty} x_m^{\frac{1}{u_2}}\right)^{u_2} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{1}{v_2}}\right)^{v_2} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{1}{w_2}}\right)^{w_2},
\end{aligned}$$

where $K_2 = (\sum f(m))^{\alpha} (\sum g(n))^{\beta} \cdots (\sum h(s))^{\gamma}$.

It follows from Lemma 3, that

$$\begin{aligned}
&\sum a_{mn\cdots s}^{\lambda} (f^{\alpha}(m) g^{\beta}(n) \cdots h^{\gamma}(s))^{k_2} x_m y_n \cdots z_s \\
\leq&K^{\lambda}\left(\frac{p_1}{c}, \frac{p_2}{c}, \ldots, \frac{p_r}{c}\right) \left(\sum_{m=1}^{\infty} f(m)\right)^{\alpha k_2} \left(\sum_{n=1}^{\infty} g(n)\right)^{\beta k_2} \cdots \left(\sum_{s=1}^{\infty} h(s)\right)^{\gamma k_2} \\
&\left(\sum_{m=1}^{\infty} x_m{}^{\frac{q_1}{c}}\right)^{\frac{c}{q_1}} \left(\sum_{n=1}^{\infty} y_n{}^{\frac{q_2}{c}}\right)^{\frac{c}{q_2}} \cdots \left(\sum_{s=1}^{\infty} z_s{}^{\frac{q_r}{c}}\right)^{\frac{c}{q_r}},
\end{aligned}$$

where $\frac{1}{\underline{p}_i} = \frac{1}{r\lambda}\left(\frac{r}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j} + \lambda\right)$, $i = 1, 2, \ldots, r$, $k_2 = (\sum_{i=1}^{r} \frac{1}{q_i} - \lambda)/r$.

This completes the proof of Theorem 3.

Theorem 4. *Let $c \in (0, 1], \alpha \geq 0, \beta \geq 0, \ldots, \gamma \geq 0$ such that $\alpha + c = 1, \beta + c = 1, \ldots, \gamma + c = 1$ and $w(x), r(y), \ldots, l(z)$ are nonnegative integrable functions.*

Let $k(x, y, \ldots, z) \geq 0$ and suppose that for every $p_1 > 1, p_2 > 1, \ldots, p_r > 1$ s.t. $\sum_{j=1}^{r} p_i^{-1} = 1$ and $f(x) \geq 0$, $g(y) \geq 0, \ldots, h(z) \geq 0$ the following inequality is valid

$$\begin{aligned}
&\int\int \cdots \int k(x, y, \ldots, z) f(x) g(y) \cdots h(z) dx dy \cdots dz \\
\leq& K\left(\frac{p_1}{c}, \frac{p_2}{c}, \ldots, \frac{p_r}{c}\right) \left(\int f^{\frac{p_1}{c}}(x) dx\right)^{\frac{c}{p_1}} \left(\int g^{\frac{p_2}{c}}(y) dy\right)^{\frac{c}{p_2}} \cdots \left(\int h^{\frac{p_r}{c}}(z) dz\right)^{\frac{c}{p_r}}.
\end{aligned}$$

If $q_i > 1, i = 1, 2, \ldots, r$ and $0 < \lambda < 1, 1 + \frac{r-1}{q_i} > \sum_{j=1}^{r} \frac{1}{q_j} > 1, i = 1, 2, \ldots, r.,$

$(r-1)\lambda = \sum_{j=1}^{r} \frac{1}{q_j'}$, then

$$\int\int\cdots\int k^{\lambda}(x,y,\ldots,z)(w^{\alpha}(x)r^{\beta}(y)\cdots l^{\gamma}(z))^{1-\lambda} f(x)g(y)\cdots h(z)dxdy\cdots dz$$
$$\leq K^{\lambda}(\frac{\overline{p}_1}{c},\frac{\overline{p}_2}{c},\ldots,\frac{\overline{p}_r}{c})$$
$$\left(\int w(x)dx\right)^{\alpha(1-\lambda)}\left(\int r(y)dy\right)^{\beta(1-\lambda)}\cdots\left(\int l(z)dz\right)^{\gamma(1-\lambda)}$$
$$\left(\int f^{\frac{q_1}{c}}(x)dx\right)^{\frac{c}{q_1}}\left(\int g^{\frac{q_2}{c}}(y)dy\right)^{\frac{c}{q_2}}\cdots\left(\int h^{\frac{q_r}{c}}(z)dz\right)^{\frac{c}{q_r}},$$

where $\overline{p}_i = \lambda(\lambda - 1 + \frac{1}{q_i})^{-1}, i = 1,2,\ldots,r$.

Proof. We need the following

Lemma 4. *Let $k(x,y,\ldots,z) \geq 0$, $m(x,y,\ldots,z) \geq 0$, $u_1 > 0$, $v_1 > 0,\ldots,w_1 > 0, u_2 > 0, v_2 > 0,\ldots,w_2 > 0$ and suppose that for every nonnegative $f(x),g(y),\ldots$, $h(z)$ the following inequalities are valid*

$$\int\int\cdots\int k(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz \leq K_1$$
$$\left(\int f(x)^{\frac{1}{u_1}}dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v_1}}dy\right)^{v_1}\cdots\left(\int h(z)^{\frac{1}{w_1}}dz\right)^{w_1} \quad (8)$$
$$\int\int\cdots\int m(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz \leq K_2$$
$$\left(\int f(x)^{\frac{1}{u_2}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v_2}}dy\right)^{v_2}\cdots\left(\int h(z)^{\frac{1}{w_2}}dz\right)^{w_2} \quad (9).$$

Futher, let $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 = 1$, $u = k_1u_1 + k_2u_2$, $v = k_1v_1 + k_2v_2,\ldots,w = k_1w_1 + k_2w_2$. Then

$$\int\int\cdots\int k^{k_1}(x,y,\ldots,z)m^{k_2}(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz \leq K_1^{k_1}K_2^{k_2}$$
$$\left(\int f(x)^{\frac{1}{u}}dx\right)^{u}\left(\int g(y)^{\frac{1}{v}}dy\right)^{v}\cdots\left(\int h(z)^{\frac{1}{w}}dz\right)^{w}.$$

Proof of Lemma 4. By Hölder inequality, since $k_1 + k_2 = 1$, it follwos

$$\int\int\cdots\int k^{k_1}(x,y,\ldots,z)m^{k_2}(x,y,\ldots,z)f(x)g(y)\cdots h(z)dxdy\cdots dz$$
$$=\int\int\cdots\int\left(kf^{\frac{u_1}{u}}g^{\frac{v_1}{v}}\cdots h^{\frac{w_1}{w}}\right)^{k_1}\left(mf^{\frac{u_2}{u}}g^{\frac{v_2}{v}}\cdots h^{\frac{w_2}{w}}\right)^{k_2}dxdy\cdots dz$$

$$\leq \left(\int\int\cdots\int k(x,y,\ldots,z) f^{\frac{u_1}{u}} g^{\frac{v_1}{v}} \cdots h^{\frac{w_1}{w}} dxdy\cdots d_z\right)^{k_1}$$
$$\left(\int\int\cdots\int m(x,y,\ldots,z) f^{\frac{u_2}{u}} g^{\frac{v_2}{v}} \cdots h^{\frac{w_2}{w}} dxdy\cdots d_z\right)^{k_2}. \tag{10}$$

As consequences of (8) and (9), we have

$$\int\int\cdots\int k(x,y,\ldots,z) f^{\frac{u_1}{u}}(x) g^{\frac{v_1}{v}}(y) \cdots h^{\frac{w_1}{w}}(z) dxdy\cdots dz$$
$$\leq K_1 \left(\int f(x)^{\frac{1}{u}} dx\right)^{u_1} \left(\int g(y)^{\frac{1}{v}} dy\right)^{v_1} \cdots \left(\int h(z)^{\frac{1}{w}} dz\right)^{w_1}$$
$$\int\int\cdots\int m(x,y,\ldots,z) f^{\frac{u_2}{u}}(x) g^{\frac{v_2}{v}}(y) \cdots h^{\frac{w_2}{w}}(z) dxdy\cdots dz$$
$$\leq K_2 \left(\int f(x)^{\frac{1}{u}} dx\right)^{u_2} \left(\int g(y)^{\frac{1}{v}} dy\right)^{v_2} \cdots \left(\int h(z)^{\frac{1}{w}} dz\right)^{w_2}.$$

Substituting these inequalities into (10), we obtain

$$\int\int\cdots\int k^{k_1}(x,y,\ldots,z) m^{k_2}(x,y,\ldots,z) f(x) g(y) \cdots h(z) dxdy\cdots dz$$
$$\leq \left(K_1 \left(\int f(x)^{\frac{1}{u}} dx\right)^{u_1} \left(\int g(y)^{\frac{1}{v}} dy\right)^{v_1} \cdots \left(\int h(z)^{\frac{1}{w}} dz\right)^{w_1}\right)^{k_1}$$
$$\left(K_2 \left(\int f(x)^{\frac{1}{u}} dx\right)^{u_2} \left(\int g(y)^{\frac{1}{v}} dy\right)^{v_2} \cdots \left(\int h(z)^{\frac{1}{w}} dz\right)^{w_2}\right)^{k_2}$$
$$= K_1^{k_1} K_2^{k_2} \left(\int f^{\frac{1}{u}}(x) dx\right)^{k_1u_1+k_2u_2} \left(\int g^{\frac{1}{v}}(y) dy\right)^{k_1v_1+k_2v_2}$$
$$\cdots \left(\int h^{\frac{1}{w}}(z) dz\right)^{k_1w_1+k_2w_2}$$
$$= K_1^{k_1} K_2^{k_2} \left(\int f(x)^{\frac{1}{u}} dx\right)^{u} \left(\int g(y)^{\frac{1}{v}} dy\right)^{v} \cdots \left(\int h(z)^{\frac{1}{w}} dz\right)^{w}.$$

Proof of Theorem 4. If we set $m(x,y,\ldots,z) = w^{\alpha}(x) r^{\beta}(y) \cdots l^{\gamma}(z), u_1 = \frac{c}{p_1}, v_1 = \frac{c}{p_2}, \ldots, w_1 = \frac{c}{p_r}$ s.t. $\sum_{i=1}^{r} \frac{1}{p_i} = 1, u_2 = v_2 = \ldots = w_2 = c, k_1 = \lambda, u = \frac{c}{q_1}, v = \frac{c}{q_2}, \ldots, w = \frac{c}{q_r}$. Since

$$\begin{cases} u = k_1u_1 + k_2u_2 \\ v = k_1v_1 + k_2v_2 \\ \vdots \\ w = k_1w_1 + k_2w_2, \end{cases}$$

we have

$$\begin{cases} \frac{1}{q_1} = \lambda \frac{1}{\overline{p}_1} + k_2 \\ \frac{1}{q_2} = \lambda \frac{1}{\overline{p}_2} + k_2 \\ \vdots \\ \frac{1}{q_r} = \lambda \frac{1}{\overline{p}_r} + k_2 \end{cases}$$

So that $k_2 = 1 - \lambda$ and $\frac{1}{\overline{p}_i} = \frac{1}{\lambda}\left(\lambda - 1 + \frac{1}{q_i}\right), i = 1, 2, \ldots, r$

The conditions of Lemma 4 are fulfield, since we have $k_2 > 0, k_1 + k_2 = 1$

Now to check $\overline{p}_i > 1, i = 1, 2, \ldots, r$ where $\overline{p}_i = \lambda(\lambda - 1 + \frac{1}{q_i})^{-1}$. We observe

$$\begin{aligned} \langle i \rangle \quad \lambda - 1 + \frac{1}{q_i} &= \frac{\sum_{j=1}^{r} \frac{1}{q_j{}'}}{r-1} - 1 + \frac{1}{q_i} \\ &= \frac{\sum_{j=1}^{r} \frac{1}{q_i{}'} - (r-1) + \frac{r-1}{q_i}}{r-1} \\ &= \frac{r - \sum_{j=1}^{r} \frac{1}{q_j} - r + 1 + \frac{r-1}{q_i}}{r-1} \\ &= \frac{1 + \frac{r-1}{q_i} - \sum_{j=1}^{r} \frac{1}{q_j}}{r-1} > 0 \end{aligned}$$

$$\text{and } \langle ii \rangle \qquad \lambda - \left(\lambda - 1 + \frac{1}{q_i}\right) = 1 - \frac{1}{q_i} > 0.$$

It follows from $\langle i \rangle$ and $\langle ii \rangle$, that $\overline{p}_i > 1, i = 1, 2, \ldots, r$.

Now we have:

$$\begin{aligned} &\int\int \cdots \int k(x, y, \ldots, z) f(x) g(y) \cdots h(z) dx dy \cdots dz \\ \leq &K\left(\frac{\overline{p}_1}{c}, \frac{\overline{p}_2}{c}, \ldots, \frac{\overline{p}_r}{c}\right)\left(\int f^{\frac{\overline{p}_1}{c}}(x) dx\right)^{\frac{c}{\overline{p}_1}}\left(\int g^{\frac{\overline{p}_2}{c}}(y) dy\right)^{\frac{c}{\overline{p}_2}} \cdots \left(\int h^{\frac{\overline{p}_r}{c}}(z) dz\right)^{\frac{c}{\overline{p}_r}} \\ = &K_1\left(\int f(x)^{\frac{1}{u_1}} dx\right)^{u_1}\left(\int g(y)^{\frac{1}{v_1}} dy\right)^{v_1}\left(\int h(z)^{\frac{1}{w_1}} dz\right)^{w_1}. \end{aligned}$$

where $K_1 = K(\frac{\overline{p}_1}{c}, \frac{\overline{p}_2}{c}, \ldots, \frac{\overline{p}_r}{c})$

Futhermore, Since $\alpha + c = 1, \beta + c = 1, \ldots, \gamma + c = 1$, it follwos from Hölder inequality that:

$$\begin{aligned}
&\int\int\cdots\int w^{\alpha}(x)r^{\beta}(y)\cdots l^{\gamma}(z)f(x)g(y)\cdots h(z)dxdy\cdots dz\\
\le&\left(\int w^{\alpha}(x)\left(f^{\frac{1}{c}}(x)\right)^{c}dx\right)\left(\int r^{\beta}(y)\left(g^{\frac{1}{c}}(y)\right)^{c}dy\right)\cdots\left(\int l^{\gamma}(z)\left(h^{\frac{1}{c}}(z)\right)^{c}dz\right)\\
\le&\left(\int w(x)dx\right)^{\alpha}\left(\int f^{\frac{1}{c}}(x)dx\right)^{c}\left(\int r(y)dy\right)^{\beta}\left(\int g^{\frac{1}{c}}(y)dy\right)^{c}\\
&\cdots\left(\int l(z)dz\right)^{\gamma}\left(\int h^{\frac{1}{c}}(z)dz\right)^{c}\\
=&K_2\left(\int f(x)^{\frac{1}{u_2}}dx\right)^{u_2}\left(\int g(y)^{\frac{1}{v_2}}dy\right)^{v_2}\cdots\left(\int h(z)^{\frac{1}{w_2}}dz\right)^{w_2},
\end{aligned}$$

where $K_2=(\int w(x)dx)^{\alpha}(\int r(y)dy)^{\beta}\cdots(\int l(z)dz)^{\gamma}$.

It follows from Lemma 4 that

$$\begin{aligned}
&\int\int\cdots\int k^{\lambda}(x,y,\ldots,z)(w^{\alpha}(x)r^{\beta}(y)\cdots l^{\gamma}(z))^{1-\lambda}f(x)g(y)\cdots h(z)dxdy\cdots dz\\
\le&K^{\lambda}\left(\frac{\overline{p}_1}{c},\frac{\overline{p}_2}{c},\ldots\frac{\overline{p}_r}{c}\right)\left(\int w(x)dx\right)^{\alpha(1-\lambda)}\left(\int r(y)dy\right)^{\beta(1-\lambda)}\cdots\left(\int l(z)dz\right)^{\gamma(1-\lambda)}\\
&\left(\int f^{\frac{q_1}{c}}(x)dx\right)^{\frac{c}{q_1}}\left(\int g^{\frac{q_2}{c}}(y)dy\right)^{\frac{c}{q_2}}\cdots\left(\int h^{\frac{q_r}{c}}(z)dz\right)^{\frac{c}{q_r}}
\end{aligned}$$

where $\overline{p}_i=\lambda(\lambda-1+\frac{1}{q_i})^{-1}$, $i=1,2,\ldots,r$.

This completes the proof of Theorem 4.

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