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SOME NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of variational inequalities. Using the auxiliary principle technique, we prove the existence of a solution of this class of variational inequalities and suggest a new and novel iterative algorithm. Several special cases, which can be obtained from the main results, are also discussed.

1. Introduction and Formulation

An elegant theory of variational inequalities has been developed since the early sixties, which has greatly stimulated the research in pure and applied sciences. In the last thirty years remarkable progress has been made in the field of variational inequalities. Variational inequalities arise in models for a large number of mathematical, physical, regional, engineering and other problems. The theory of variational inequalities has led to exciting and important contributions to pure and applied sciences which includes work on differential equations, contact problems in elasticity, fluid flow through porous media, general equilibrium problems in economics and transportation, unilateral, obstacle, moving and free boundary problems, see [1-22]. Inspired and motivated by the recent research work going on in this field, we introduce and consider some new classes of variational inequalities. We remark that the projection method and its variant form cannot be applied to study the existence of a solution of these new variational inequalities. This fact motivated us to use the auxiliary principle technique of Glowinski, Lions and Tremolieres [6] and Noor [13,14,16-18] to study the problem of the existence of these variational inequalities. This technique deals with an auxiliary variational inequality problem and proving that the solution of the auxiliary problem is the solution of the original variational inequality problem. This technique is quite general and flexible. In recent years, the auxiliary principle technique has being used to suggest unified descent algorithms for solving variational inequalities. It has been shown by Noor [17] that this

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technique can be used to formulate the equivalent differentiable optimization problems for variational inequalities. In this paper, we use this technique to suggest an iterative algorithm for variational inequalities.

To be more precise, let H be a real Hilbert space on which inner product and norm are denoted by $<.,.>$ and $\|\cdot\|$ respectively. Let K be a nonempty closed convex set in H. Given $T, g, A : H \to H$ continuous operators, we consider the problem of finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu - A(u), v - g(u) > + b(u, v) - b(u, g(u)) \geq 0, \qquad \text{for all } v \epsilon K, \tag{1.1}$$

where the form $b(.,.) : H \times H \to R$ is non-differentiable and satisfies the following:
(i) $b(u, v)$ is linear in the first argument.
(ii) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(u, v) \leq \gamma \|u\| \|v\|, \qquad \text{for all } u, v \epsilon H. \tag{1.2}$$

(iii) $b(u, v) - b(u, w) \leq b(u, v - w), \qquad$ for all $u, v, w \epsilon H$.

The inequalities of the type (1.1) are called the strongly nonlinear mixed variational inequalities. We now discuss some special cases.

I. Note that if $g \equiv I$, the identity operator, then problem (1.1) is equivalent to finding $u \epsilon K$ such that

$$< Tu - A(u), v - u > + b(u, v) - b(u, u) \geq 0, \qquad \text{for all } v \epsilon K, \tag{1.3}$$

a problem considered and studied by Noor [13].

II. If $b(u, v) \equiv j(v)$ is a convex, lower semi- continuous, proper and non-differentiable functional, then the problem (1.1) is equivalent to finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu - A(u), v - g(u) > + j(v) - j(g(u)) \geq 0, \qquad \text{for all } v \epsilon K, \tag{1.4}$$

which is called the mixed variational inequality problem and appears to be new.

III. If $b(v, u) \equiv 0$, and $A(u) \equiv 0$, then problem (1.1) reduces to the problem of finding $u \epsilon H$ such that $g(u) \epsilon K$ and

$$< Tu, v - g(u) > \geq 0, \qquad \text{for all } v \epsilon K, \tag{1.5}$$

a problem introduced by Oettli[20], Isac[8] and Noor[12] in different contexts and applications.

IV. If $b(u, v) \equiv 0$, $K^* = \{u \epsilon H; < u, v > \geq 0 \text{ for all } v \epsilon K\}$ is a polar cone of the convex cone K in H, then problem (1.1) is equivalent to finding $u \epsilon H$ such that

$$g(u) \epsilon K, \quad Tu - A(u) \epsilon K^* \quad \text{and} \quad < Tu - A(u), g(u) > = 0, \tag{1.6}$$

which is known as the general nonlinear complementarity problem. The problem (1.6) is quite general and includes many previously known classes of linear and nonlinear complementarity problems as special cases.

For appropriate and suitable choice of the operators T, A, g, the form $b(.,.)$ and the convex set K, one can obtain a number of known and unknown classes of variational inequalities and complementarity problems as special cases from the problem (1.1). In brief, it is clear that the problem (1.1) is the most general and unifying one, which is one of the main motivation of this paper.

Definition 1.1 A mapping $T : H \to H$ is said to be
(a) *Strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$< Tu - Tv, u - v > \geq \alpha \|u - v\|^2, \qquad \text{for all } u, v \epsilon H.$$

(b) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \qquad \text{for all } u, v \epsilon H.$$

In particular, it follows that $\alpha \leq \beta$.

Concerning the unique solution of the variational inequality (1.1), we need the following assumptions.

Condition M. We assume that $k(\gamma + \mu) < \alpha$, where $\alpha > 0$ is the strongly monotonicity constant of the nonlinear operator T, γ is the boundedness constant of the form $b(u, v), \mu > 0$ is the Lipschitz continuity constant of the operator A and $k = \frac{\eta}{\xi} \neq 0$. Here $\eta > 0$ and $\xi > 0$ are the strongly monotonicity and Lipschitz continuity constants of the operator g.

Condition L. We assume that $\nu < \alpha$, where $\alpha > 0$ is the strongly monotonicity constant of T and $\nu = \beta\sqrt{1 - 2\eta + \xi^2} + \xi(\gamma + \mu)$. Here $\eta > 0$ is the strongly monotonicity constant of the operator g, $\xi > 0$, $\mu > 0$ are the Lipschitz continuity constants of the operators g and A respectively and $\gamma > 0$ is the boundedness constant of the form $b(.,.)$.

We like to point out that if the operator $g \equiv I$, the identity operator, then $k = 1 = \xi = \eta$. Consequently the condition M and condition L are exactly the same condition N in Noor [13]. This shows that the condition M and condition L are compatible with condition N.

2. Main Results

In this section, we prove the existence of a unique solution of the mixed variational inequality problem (1.1) by using the auxiliary principle technique and this is the main motivation of our next result.

Theorem 2.1. *Let the operators $T, g : H \to H$ be both strongly monotone Lipschitz continuous and the form $b(.,.) : H \times H \to H$ satisfy the conditions* (i)-(iii). *Let the operator $A : H \to H$ be a Lipschitz continuous with constant $\mu > 0$. If the condition M and condition L hold, then there exists a unique solution $u \epsilon H$ such that $g(u) \epsilon K$ and* (1.1) *holds.*

Proof.
(a) *Uniqueness.* Let $u_1, u_2 \epsilon H$, $u_1 \neq u_2$ be two solutions of the variational inequality (1.1). Then

$$< Tu_1 - A(u_1), v - g(u_1) > + b(u_1, v) - b(u_1, g(u_1)) \geq 0 \quad \text{for all } v \epsilon K, \tag{2.1}$$

and

$$< Tu_2 - A(u_2), v - g(u_2) > + b(u_2, v) - b(u_2, g(u_2)) \geq 0 \quad \text{for all } v \epsilon K. \tag{2.2}$$

Taking $v = g(u_2)$ in (2.1) and $v = g(u_1)$ in (2.2) and adding the resultant inequalities, we obtain

$$< Tu_1 - Tu_2, g(u_1) - g(u_2) > \leq b(u_1 - u_2, g(u_2) - g(u_1)) + < A(u_1) - A(u_2), g(u_1) - g(u_2) >,$$

which can be written as

$$\begin{aligned} < Tu_1 - Tu_2, u_1 - u_2 > \leq & < Tu_1 - Tu_2, u_1 - u_2 - (g(u_1) - g(u_2)) > \\ & + b(u_1 - u_2, g(u_2) - g(u_1)) \\ & + < A(u_1) - A(u_2), g(u_1) - g(u_2) > . \end{aligned}$$

Using the strongly monotonicity of T, (1.2) and applying the Cauchy- Schwartz inequality, we have

$$\begin{aligned} \alpha\|u_1 - u_2\|^2 \leq & < Tu_1 - Tu_2, u_1 - u_2 > \\ \leq & \|Tu_1 - Tu_2\| \|u_1 - u_2 - (g(u_1) - g(u_2))\| + \gamma\|u_1 - u_2\| \|g(u_1) - g(u_2)\| \\ & + \|A(u_1) - A(u_2)\| \|g(u_1) - g(u_2)\| \\ \leq & \|u_1 - u_2\| \|u_1 - u_2 - (g(u_1 - g(u_2))\| + \gamma\xi\|u_1 - u_2\|^2 \\ & + \mu\xi\|u_1 - u_2\|^2, \end{aligned} \tag{2.3}$$

where $\beta > 0, \mu > 0$ and $\xi > 0$ are the Lipschitz continuity constants of the operators T, A and g respectively.

Since g is a strongly monotonicity Lipschitz continuous operator, so

$$\begin{aligned} \|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 = & \|u_1 - u_2\|^2 - 2 < g(u_1) - g(u_2), u_1 - u_2 > \\ & + \|g(u_1) - g(u_2)\|^2 \\ \leq & (1 - 2\eta + \xi^2)\|u_1 - u_2\|^2. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned}\alpha\|u_1-u_2\|^2 \le& \{\beta\sqrt{1-2\eta+\xi^2}+\xi(\gamma+\mu)\}\|u_1-u_2\|^2\\ =&\nu\|u_1-u_2\|^2, \quad \text{by condition L.}\end{aligned}$$

Thus

$$(\alpha-\nu)\|u_1-u_2\|^2 \le 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution, since $\nu < \alpha$ by condition L.

(b) *Existence.* We now use the auxiliary principle technique to prove the existence of a solution of (1.1) using the ideas of Glowinski, Lions and Tremolieres [6] and Noor [13,16]. For a given $u\epsilon H$ such that $g(u)\epsilon K$, we consider the problem of finding a unique $w\epsilon H$ such that $g(w)\epsilon K$, (see[6]), satisfying the auxiliary variational inequality

$$\begin{aligned}< w, v-g(w) > +\rho b(u,v) - \rho b(u, g(w)) \ge& < u, v-g(w) >\\ &-\rho < Tu - Au, v-g(w) >,\end{aligned} \tag{2.5}$$

for all $v\epsilon K$, where $\rho > 0$ is a constant.

Let w_1, w_2 be two solutions of (2.5) related to $u_1, u_2\epsilon H$ respectively. It is enough to show that the mapping $u \to w$ has a fixed point belonging to H satisfying (1.1). In other words, it is sufficient to show that for well chosen $\rho > 0$,

$$\|w_1-w_2\| \le \theta\|u_1-u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = g(w_2)$ (respectively $g(w_1)$) in (2.5) related to u_1 (respectively u_2), we have

$$\begin{aligned}&< w_1, g(w_2)-g(w_1) > +\rho b(u_1, g(w_2)) - \rho b(u_1, g(w_1))\\ &\ge < u_1, g(w_2)-g(w_1) > -\rho < Tu_1 - A(u_1), g(w_2)-g(w_1) >\end{aligned}$$

and

$$\begin{aligned}&< w_2, g(w_1)-g(w_2) > +\rho b(u_2, g(w_1)) - \rho b(u_2, g(w_2))\\ &\ge < u_2, g(w_1)-g(w_2) > -\rho < Tu_2 - A(u_2), g(w_1)-g(w_2) > .\end{aligned}$$

Adding these inequalities and using (iii), we have

$$\begin{aligned}&< w_1-w_2, g(w_1)-g(w_2) >\\ &\le < u_1-u_2, g(w_1)-g(w_2) > +\rho b(u_1-u_2, g(w_2)-g(w_1))\\ &\quad -\rho < Tu_1-Tu_2, g(w_1)-g(w_2) > +\rho < A(u_1)-A(u_2), g(w_1)-g(w_2) >\\ &= < u_1-u_2-\rho(Tu_1-Tu_2), g(w_1)-g(w_2) > +\rho b(u_1-u_2, g(w_2)-g(w_1))\\ &\quad +\rho < A(u_1)-A(u_2), g(w_1)-g(w_2) >,\end{aligned}$$

from which using (1.2) and the strongly monotonicity of g, we obtain

$$\begin{aligned}\eta\|w_1 - w_2\|^2 \leq& \{\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\| \\ &+ \rho\gamma\|u_1 - u_2\| + \rho\|A(u_1) - A(u_2)\|\}\|g(w_1) - g(w_2)\| \\ \leq& \xi\{\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\| + \rho\gamma\|u_1 - u_2\| \\ &+ \mu\|u_1 - u^2\|\}\|w_1 - w_2\|, \end{aligned} \tag{2.6}$$

where $\eta > 0$ and $\xi > 0$ are the strongly monotonicity and Lipschitz continuity constants of the operator g and $\mu > 0$ is the Lipschitz continuity constant of A.

Since T is a strongly monotone Lipschitz continuous operator, so

$$\begin{aligned}&\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \\ \leq& \|u_1 - u_2\|^2 - 2\rho < u_1 - u_2, Tu_1 - Tu_2 > + \rho^2\|Tu_1 - Tu_2\|^2 \\ \leq& (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$\begin{aligned}\|w_1 - w_2\| \leq& \left\{ \frac{\rho(\gamma + \mu) + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}}{k} \right\} \|u_1 - u_2\|, \quad \text{where} \quad k = \frac{\eta}{\xi} \neq 0. \\ =& \theta\|u_1 - u_2\|, \end{aligned}$$

with $\theta = \dfrac{\rho(\gamma + \mu) + t(\rho)}{k}$ and $t(\rho) = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}$.

We have to show that $\theta < 1$. It is clear that $t(\rho)$ assumes its minimum value for $\bar{\rho} = \frac{\alpha}{\beta^2}$ with $t(\bar{\rho}) = \sqrt{1 - (\frac{\alpha^2}{\beta^2})}$. For $\rho = \bar{\rho}, \rho(\gamma + \mu) + t(\bar{\rho}) < k$ implies that $\rho(\gamma + \mu) < k$. Thus it follows that $\theta < 1$ for all ρ with

$$| \rho - \frac{\alpha - k(\gamma + \mu)}{\beta^2 - (\gamma + \mu)^2} | < \frac{\sqrt{(\alpha - k(\gamma + \mu))^2 - (\beta^2 - (\gamma + \mu)^2)(1 - k^2)}}{\beta^2 - (\gamma + \mu)^2},$$

and $\rho(\gamma + \mu) < k$ by condition M, showing that the mapping $u \to w$ defined by (2.5) has a fixed point, which is the solution of (1.1), the required result.

Remark 2.1. If $g = I$ and $A(u) = 0$, the identity operator, then problem (2.5) is equivalent to finding $w \epsilon H$ for a given $u \epsilon H$ such that

$$< w, v - w > + \rho b(u, v) - \rho b(u, w) \geq < u, v - w > - \rho < Tu, v - w >, \tag{2.8}$$

for all $v \epsilon K$ and $\rho > 0$, is a constant. From the proof of Theorem 2.1, we see that $k = 1$ and $\theta = \rho\gamma + t(\rho) < 1$ for $0 < \rho < 2\frac{\alpha - \gamma}{\beta^2 - \gamma^2}, \quad \gamma < \alpha$ and $\rho\gamma < 1$, so the mapping $u \to w$ defined by (2.4) has a fixed point, which is the solution of the variational inequality (1.3) studied by Kikuchi and Oden [9] in elasticity. Similarly for appropriate choice of the operators T, g, A, the form $b(u, v)$ and the convex set K, we can apply Theorem 2.1 to prove the existence of a unique solution for various classes of variational inequalities studied previously.

Remark 2.2. It is clear that if $w = u$, then w is the solution of the variational inequality (1.1). This observation enables us to suggest an iterative algorithm for finding the approximate solution of the variational inequality (1.1) and its various special cases.

Algorithm 2.1.
(a) At $n = 0$, start with some initial w_0.
(b) At step n, solve the auxiliary problem (2.5) with $u = w_n$. Let w_{n+1} denote the solution of the problem (2.5).
(c) For given $\varepsilon > 0$, if $\|w_{n+1} - w_n\| \leq \varepsilon$, stop. Otherwise repeat (b).

Remark 2.3. It is worth mentioning that many previously known methods including projection techniques and its variant forms, linear approximation, relaxation, descent and Newton algorithms that have been proposed for solving various classes of variational inequalities and complementarity problems can be derived as special cases from the auxiliary principle technique, see Noor [16,17,18]. It is known [17] that the auxiliary principle technique can be used to formulate the equivalent (non) differentiable optimization problems for variational inequalities. For illustration purpose, we consider a special case of the mixed variational inequality (1.1). Let $g \equiv I$, the identity operator and $b(u, v) \equiv 0$, then for a given $u \epsilon K$, the auxiliary variational inequality problem (2.5) is equivalent to finding a unique $w \epsilon K$ such that

$$< w, v - w > \geq < u, v - w > - \rho < Tu - A(u), v - w >, \quad \text{for all } v \epsilon K, \tag{2.9}$$

where $\rho > 0$ is a constant.

It is obvious that the problem (2.9) is equivalent to finding the minimum of the functional $I[w]$ on the convex set K in H, where

$$I[w] = \frac{1}{2} < w - u, w - u > + \rho < Tu - A(u), w - u >, \tag{2.10}$$

which is an auxiliary quadratic differential functional associated with the problem (2.9). Using the technique of Fukushima [5], one can prove that the variational inequality problem of finding $u \epsilon K$ such that

$$< Tu - A(u), v - u > \geq 0, \quad \text{for all } v \epsilon K, \tag{2.11}$$

is equivalent to finding the minimum of the functional $F[u]$ on K in H, where

$$F[u] = \frac{1}{2} < u - w(u), w(u) - u > + < Tu - A(u), u - w(u) >, \tag{2.12}$$

where $w = w(u)$ is the solution of the auxiliary variational inequality (2.9). The functional $F[u]$ defined by (2.12) is known as the gap (merit) function associated the variational inequality (2.11). These gap (merit) functions can be used to develop general framework for descent and Newton methods with line search to solve the variational inequalities of the type (2.11). For recent developments in this direction, see Larsson and Patriksson [11], Zhu and Marcotte [22] and Noor [17,18].

Using the ideas and techniques of Noor [16], we can propose and analyze a general algorithm. For a given $u \epsilon K$, we introduce the following general auxiliary problem of finding the minimum of the functional $N[w]$ on K in H, where

$$\begin{aligned} N[w] =& E(w) - < E'(u), u > + \rho < Tu - A(u), w >, \\ =& E(w) - < E'(u), w - u > + \rho < Tu - A(u), w - u > \\ &+ \rho < Tu - A(u), u >, - < E'(u), u > \\ =& E(w) - E(u) - < E'(u), w - v > + \rho < Tu - A(u), w - u > + E(u) \\ &+ \rho < Tu - A(u), u > - < E'(u), u > . \end{aligned}$$

This implies that for a given $u \epsilon K$, we consider the minimum of the auxiliary functional $M[w]$ on K in H, where

$$M[w] = E(w) - E(u) - < E'(u), w - n > + \rho < Tu - A(w), w - u > .$$

Here $E(w)$ is a differentiable convex function. Thus we can associate to (2.11), the equivalent optimization problem

$$\max\{M[w], w \epsilon K\},$$

which is called the variational principle for the variational inequality (2.11), see Blum and Oettli [3] for more details. This shows that by a suitable choice of the auxiliary problem, one can suggest a large number of equivalent differentiable optimization problems for various classes of variational inequalities and complementarity problems. It is an open problem to find the equivalent differentiable optimization problem for the mixed variational inequality (1.1). This needs further research. We remark that if the convex set K also depends on the solution itself implicitly or explicitly, then the mixed variational inequality (1.1) is known as the mixed quasi variational inequality. Extending the auxiliary principle technique for quasi variational inequalities is still an open problem and this constitute an important and interesting area of the future research.

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