TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 2, Summer 1995

A GENERALIZATION OF CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

M. K. AOUF AND A. SHAMANDY

Abstract. We introduce the subclass $T^*(A,B,n,\alpha)$ $(-1 \le A < B \le 1,\ 0 < B \le 1,\ n \ge 0$, and $0 \le \alpha < 1)$ of analytic functions with negative coefficients by the operator D^n. Coefficient estimates, distortion theorems, closure theorems and radii of close-to-convexety, starlikeness and convexity for the class $T^*(A,B,n,\alpha)$ are determined. We also prove results involving the modified Hadamard product of two functions associated with the class $T^*(A,B,n,\alpha)$. Also we obtain several interesting distortion theorems for certain fractional operators of functions in the class $T^*(A,B,n,\alpha)$. Also we obtain class perserving integral operator of the form

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -1$$

for the class $T^*(A,B,n,\alpha)$. Conversely when $F(z) \in T^*(A,B,n,\alpha)$, radius of univalence of $f(z)$ defined by the above equation is obtained.

1. Introduction

let A_1 denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$, and let S denote the subclass of A_1 consisting of analytic and univalent functions $f(z)$ in the unit disc U. We use Ω to denote the class of analytic functions $w(z)$ in U satisfies the conditions $w(0) = 0$ and $|w(z)| \le |z|$ for $z \in U$.

Received September 3, 1993; revised September 26, 1994.

1991 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, convex, modified Hadamard procuct, fractional calculus.

For a function $f(z)$ in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N} = \{1, 2, \ldots\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [7]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to S is in the class $S(A, B, n, \alpha)$ $(-1 \leq A < B \leq 1, 0 < B \leq 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $0 \leq \alpha < 1)$ if and only if

$$\frac{D^{n+1} f(z)}{D^n f(z)} \prec \frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}, \quad z \in U. \tag{1.5}$$

Equivalently, a function $f(z)$ of S belongs to the class $S(A, B, n, \alpha)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}, \quad z \in U. \tag{1.6}$$

It is easy to see that the condition (1.6) is equivalent to

$$\left| \frac{\frac{D^{n+1} f(z)}{D^n f(z)} - 1}{B \frac{D^{n+1} f(z)}{D^n f(z)} - [B + (A - B)(1 - \alpha)]} \right| < 1, \quad z \in U. \tag{1.7}$$

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1.8}$$

Further, we define the class $T^*(A, B, n, \alpha)$ by

$$T^*(A, B, n, \alpha) = S(A, B, n, \alpha) \cap T. \tag{1.9}$$

We note that, by specializing the parameters A, B, n, and α, we obtain the following subclasses studied by various authors:

(1) $T^*(-1, 1, n, \alpha) = T(n, \alpha)$ (Hur and Oh [4]);

(2) $T^*(-1, 1, 0, \alpha) = T^*(\alpha)$ and $T^*(-1, 1, 1, \alpha) = C(\alpha)$ (Silverman [9]);

(3) $T^*(-\beta, \beta, 0, \alpha) = S^*(\alpha, \beta)$ and $T^*(-\beta, \beta, 1, \alpha) = C^*(\alpha, \beta)$ $(0 \leq \alpha < 1), (0 < \beta \leq 1)$ (Gupta and Jain [3]);

(4) $T^*(A, B, 0, 0) = T_1^*(A, B)$ and $T^*(A, B, 1, 0) = C_1(A, B)$ (Goel and Sohi [2]);

(5) $T^*(A, B, 0, \alpha) = T_1^*(A, B, \alpha)$ and $T^*(A, B, 1, \alpha) = C_1(A, B, \alpha)$ (Aouf [1]);

(6) $T^*(-\beta, \mu\beta, 0, \alpha) = S^*(\alpha, \beta, \mu)$ and $T^*(-\beta, \mu\beta, 1, \alpha) = C^*(\alpha, \beta, \mu)$ $(0 \leq \alpha < 1), (0 < \beta \leq 1)$ and $(0 \leq \mu \leq 1)$ (Owa and Aouf [6]);

(7) $T^*(-\beta, \beta, n, \alpha) = S^*(\alpha, \beta, n)$, where $S^*(\alpha, \beta, n)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$\left| \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\frac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha} \right| < \beta, \quad z \in U, \tag{1.10}$$

where $0 \leq \alpha < 1$, and $0 < \beta \leq 1$.

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.8). Then $f(z) \in T^*(A, B, n, \alpha)$ $(-1 \leq A < B \leq 1, 0 < B \leq 1, n \in \mathbb{N}_0$, and $0 \leq \alpha < 1)$ if and only if*

$$\sum_{k=2}^{\infty} C_k a_k \leq (B - A)(1 - \alpha), \tag{2.1}$$

where

$$C_k = k^n \left[(1 + B)(k - 1) + (B - A)(1 - \alpha)\right]. \tag{2.2}$$

The result is sharp.

Proof. Let $|z| = 1$, then

$$\begin{aligned}
&\left|D^{n+1}f(z) - D^n f(z)\right| - \left|BD^{n+1}f(z) - [B + (A - B)(1 - \alpha)]D^n f(z)\right| \\
&= \left| -\sum_{k=2}^{\infty} k^n (k - 1) a_k z^k \right| - \left| (B-A)(1-\alpha)z - \sum_{k=2}^{\infty} k^n \left[B(k-1) + (B-A)(1-\alpha)\right] a_k z^k \right| \\
&\leq \sum_{k=2}^{\infty} k^n \left[(1 + B)(k - 1) + (B - A)(1 - \alpha)\right] a_k - (B - A)(1 - \alpha) \leq 0.
\end{aligned}$$

Hence, by the principle of maximum modulus $f(z) \in T^*(A, B, n, \alpha)$.

Conversely, suppose that

$$\left| \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{B\frac{D^{n+1}f(z)}{D^n f(z)} - [B + (A - B)(1 - \alpha)]} \right|$$

$$= \left| \frac{-\sum_{k=2}^{\infty} k^n (k - 1) a_k z^k}{(B - A)(1 - \alpha)z - \sum_{k=2}^{\infty} k^n \left[B(k - 1) + (B - A)(1 - \alpha)\right] a_k z^k} \right| < 1, \quad z \in U.$$

Since $|Re(z)| \le |z|$ for all z, we have

$$\operatorname{Re}\left\{\frac{\sum_{k=2}^{\infty} k^n (k-1) a_k z^k}{(B-A)(1-\alpha)z - \sum_{k=2}^{\infty} k^n \left[B(k-1) + (B-A)(1-\alpha)\right] a_k z^k}\right\} < 1. \tag{2.3}$$

Choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \longrightarrow 1^-$ through real values, we obtain

$$\sum_{k=2}^{\infty} k^n (k-1) a_k \le (B-A)(1-\alpha) - \sum_{k=2}^{\infty} k^n \left[B(k-1) + (B-A)(1-\alpha)\right] a_k$$

which implies that

$$\sum_{k=2}^{\infty} k^n \left[(1+B)(k-1) + (B-A)(1-\alpha)\right] a_k \le (B-A)(1-\alpha).$$

The result is sharp for the function

$$f(z) = z - \frac{(B-A)(1-\alpha)}{C_k} z^k \quad (k \ge 2). \tag{2.4}$$

Using Theorem 1, we have the following:

Corollary 1. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $T^*(A, B, n, \alpha)$. *then we have*

$$a_k \le \frac{(B-A)(1-\alpha)}{C_k} \qquad (k \ge 2). \tag{2.5}$$

The equality in (2.5) *is attained for the function* $f(z)$ *given by* (2.4).

Corollary 2. $T^*(A, B, n+1, \alpha) \subset T^*(A, B, n, \alpha)$ *for* $-1 \le A < B \le 1$, $0 < B \le 1, n \in \mathbb{N}_0$, *and* $0 \le \alpha < 1$.

3. Distortion Theorems

Theorem 2. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $T^*(A, B, n, \alpha)$. *Then we have*

$$|z| - \frac{2^i (B-A)(1-\alpha)}{C_2}|z|^2 \le |D^i f(z)| \le |z| + \frac{2^i (B-A)(1-\alpha)}{C_2}|z|^2 \tag{3.1}$$

for $z \in U$, *where* $0 \le i \le n$. *the result is sharp.*

Proof. Note that $f(z) \in T^*(A, B, n, \alpha)$ if and only if $D^i f(z) \in T^*(A, B, n-i, \alpha)$, and that

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{3.2}$$

Using Theorem 1, we can get the result. Finally, we note that the equality in (3.1) is attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{2^i (B-A)(1-\alpha)}{C_2} z^2. \tag{3.3}$$

This completes the proof of Theorem 2.

Corollary 3. *Let the function $f(z)$ defined by (1.8) be in the class $T^*(A, B, n, \alpha)$. Then we have*

$$|z| - \frac{(B-A)(1-\alpha)}{C_2}|z|^2 \le |f(z)| \le |z| + \frac{(B-A)(1-\alpha)}{C_2}|z|^2 \tag{3.4}$$

for $z \in U$. The result is sharp for the function

$$f(z) = z - \frac{(B-A)(1-\alpha)}{C_2} z^2. \tag{3.5}$$

Proof. Taking $i = 0$ in Theorem 2, we can easily show (3.8).

Corollary 4. *Let the function $f(z)$ defined by (1.8) be in the class $T^*(A, B, n, \alpha)$. Then we have*

$$1 - \frac{2(B-A)(1-\alpha)}{C_2}|z| \le |f'(z)| \le 1 + \frac{2(B-A)(1-\alpha)}{C_2}|z| \tag{3.6}$$

for $z \in U$. The result is sharp for the function $f(z)$ given by (3.5).

Proof. Note that $Df(z) = zf'(z)$. Hence, taking $i = 1$ in Theorem 2, we have Corollary 4.

4. Closure Theorems

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by

$$f_\nu(z) = z - \sum_{k=2}^{\infty} a_{k,\nu} z^k \qquad (a_{k,\nu} \ge 0,\ \nu = 1, 2). \tag{4.1}$$

Employing the techniques used earlier by Silverman [9], Gupta and Jain [3], Hur and Oh [4] and Owa and Aouf [6], and with the aid of Theorem 1, we can prove the following:

Theorem 3. *The class $T^*(A,B,n,\alpha)$ is closed under convex linear combination.*

As a consequence of Theorem 3, there exists extreme points of the class $T^*(A,B,n,\alpha)$.

Theorem 4. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{(B-A)(1-\alpha)}{C_k} z^k \quad (k \geq 2) \tag{4.2}$$

for $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and $0 \leq \alpha < 1$, Then $f(z)$ is in the class $T^*(A,B,n,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \tag{4.3}$$

where $\lambda_k \geq 0 \quad (k \geq 1)$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Corollary 5. *The extreme points of the class $T^*(A,B,n,\alpha)$ are the functions $f_k(z)$ $(k \geq 1)$ given by Theorem 4.*

5. Modified Hadamard Products

Let the functions $f_\nu(z)$ $(\nu = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{5.1}$$

Employing the technique used earlier by Schild and Silverman [8], we can prove the following:

Theorem 5. *Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by $(4,1)$ be in the class $T^*(A,B,n,\alpha)$. Then we have*

$$f_1 * f_2(z) \in T^*(A,B,n,\beta(A,B,n,\alpha)), \tag{5.2}$$

where

$$\beta(A,B,n,\alpha) = \frac{2^n - (B-A)(2B+1-A)\left[\frac{2^n(1-\alpha)}{C_2}\right]^2}{2^n - \left[\frac{2^n(B-A)(1-\alpha)}{C_2}\right]^2}. \tag{5.3}$$

The result is sharp for the functions

$$f_\nu(z) = z - \frac{(B-A)(1-\alpha)}{C_2} z^2 \quad (\nu = 1,2). \tag{5.4}$$

Theorem 6. *Let the functions* $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (4.1) *be in the class* $T^*(A, B, n, \alpha)$. *Then the function*

$$h(z) = z - \sum_{k=2}^{\infty} \left[a_{k,1}^2 + a_{k,2}^2\right] z^k \tag{5.5}$$

belongs to the class $T^*(A, B, n, \gamma(A, B, n, \alpha))$, *where*

$$\gamma(A, B, n, \alpha) = \frac{2^n - 2(B-A)(2B+1-A)\left[\frac{2^n(1-\alpha)}{C_2}\right]^2}{2^n - 2\left[\frac{2^n(B-A)(1-\alpha)}{C_2}\right]^2}. \tag{5.6}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1,2)$ *defined by* (5.4).

6. Radii of Close-to-Convexity, Starlikeness and Convexity

By using Theorem 1, we can prove the following:

Theorem 7. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $T^*(A, B, n, \alpha)$, *then* $f(z)$ *is close-to-convex of order* $\rho(0 \le \rho < 1)$ *in* $|z| < R_1$, *where*

$$R_1 = \inf_k \left[\frac{(1-\rho)C_k}{k(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.1}$$

The result is sharp, with the extremal function $f(z)$ *given by* (2.4).

Theorem 8. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $T^*(A, B, n, \alpha)$, *then* $f(z)$ *is starlike of order* $\rho(0 \le \rho < 1)$ *in* $|z| < R_2$, *where*

$$R_2 = \inf_k \left[\frac{(1-\rho)C_k}{(k-\rho)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.2}$$

The result is sharp, with the extremal function $f(z)$ *given by* (2.4).

Corollary 6. *Let the function* $f(z)$ *defined by* (1.8) *be in the class* $T^*(A, B, n, \alpha)$, *then* $f(z)$ *is convex of order* ρ $(0 \le \rho < 1)$ *in* $|z| < R_3$, *where*

$$R_3 = \inf_k \left[\frac{(1-\rho)C_k}{k(k-\rho)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.3}$$

The result is sharp, with the extremal function $f(z)$ *given by* (2.4).

7. Integral Operators

Theorem 9. *Let the function $f(z)$ defined by (1.8) be in the class $T^*(A,B,n,\alpha)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \tag{7.1}$$

also belongs to the class $T^*(A,B,n,\alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$

where

$$b_k = (\frac{c+1}{c+k})a_k.$$

Therefore,

$$\sum_{k=2}^{\infty} C_k b_k = \sum_{k=2}^{\infty} C_k(\frac{c+1}{c+k})a_k \le \sum_{k=2}^{\infty} C_k a_k \le (B-A)(1-\alpha),$$

since $f(z) \in T^*(A,B,n,\alpha)$. Hence, by Theorem 1, $F(z) \in T^*(A,B,n,\alpha)$.

Finally by using Theorem 1, we can prove the following theorem:

Theorem 10. *Let c be a real number such that $c > -1$. If $F(z) \in T^*(A,B,n,\alpha)$, then the function $f(z)$defined by (7.1) is univalent in $|z| < R^*$, where*

$$R^* = \inf_k \left[\frac{(c+1)C_k}{k(c+k)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.2}$$

The result is sharp for the function

$$f(z) = z - \frac{(c+k)(B-A)(1-\alpha)}{(c+1)C_k} z^k. \tag{7.3}$$

8. Fractional Calculus

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa [5] and used recently by Srivastava and Owa [10].

Definition 1. The fractional integral of order λ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta \quad (\lambda > 0), \tag{8.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 2. The fractional derivative of order λ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0\le\lambda<1), \tag{8.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by

$$D_z^{n+\lambda} f(z)=\frac{d^n}{dz^n}D_z^{\lambda} f(z) \quad (0\le\lambda<1; n\in\mathbb{N}_0). \tag{8.3}$$

Employing the technique used earlier by Srivastava and Owa [10], we can prove the following:

Theorem 11. *Let the function $f(z)$ defined by (1.8) be in the class $T^*(A,B,n,\alpha)$. Then we have*

$$|D_z^{-\lambda}(D^i f(z))|\ge\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{2^{i+1}(B-A)(1-\alpha)}{(2+\lambda)C_2}|z|\right\} \tag{8.4}$$

and

$$|D_z^{-\lambda}(D^i f(z))|\le\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1+\frac{2^{i+1}(B-A)(1-\alpha)}{(2+\lambda)C_2}|z|\right\} \tag{8.5}$$

for $\lambda>0$, $0\le i\le n$, and $z\in U$. The result is sharp for the function

$$D_z^{-\lambda}(D^i f(z))=\frac{z^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1-\frac{2^{i+1}(B-A)(1-\alpha)}{(2+\lambda)C_2}z\right\} \tag{8.6}$$

or

$$(D^i f(z))=z-\frac{2^{i}(B-A)(1-\alpha)}{C_2}z^2. \tag{8.7}$$

Theorem 12. *Let the function $f(z)$ defined by (1.8) be in the class $T^*(A,B,n,\alpha)$. Then we have*

$$|D_z^{\lambda}(D^i f(z))|\ge\frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1-\frac{2^{i+1}(B-A)(1-\alpha)}{(2-\lambda)C_2}|z|\right\} \tag{8.8}$$

and

$$|D_z^{\lambda}(D^i f(z))|\le\frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1+\frac{2^{i+1}(B-A)(1-\alpha)}{(2-\lambda)C_2}|z|\right\} \tag{8.9}$$

for $0 \leq \lambda < 1, 0 \leq i \leq n-1$, *and* $z \in U$. *The result is sharp for the function* $f(z)$ *given by*

$$D_z^{\lambda}(D^i f(z)) = \frac{z^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1 - \frac{2^{i+1}(B-A)(1-\alpha)}{(2-\lambda)C_2}z\right\} \tag{8.10}$$

or by $D^i f(z)$ *given by* (8.7).

9. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [11].

Definition 4. For real numbers $\beta > 0, \gamma$ and δ, the fractional integral operator $I_{o,z}^{\beta,\gamma,\delta}$ is defined by

$$I_{o,z}^{\beta,\gamma,\delta} f(z) = \frac{z^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^z (z-t)^{\beta-1} F(\beta+\gamma, -\delta; \beta; 1-\frac{t}{z}) f(t) dt \tag{9.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}), \quad z \longrightarrow 0,$$

where

$$\varepsilon > \max(0, \gamma-\delta) - 1,$$

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{9.2}$$

where $(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & (k=0) \\ \nu(\nu+1)\cdots(\nu+k-1) & (k \in \mathbb{N}), \end{cases} \tag{9.3}$$

and the multiplicity of $(z-t)^{\beta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$.

Remark. For $\gamma = -\beta$, we note that

$$I_{o,z}^{\beta,-\beta,\delta} f(z) = D_z^{-\beta} f(z).$$

In order to prove our results for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [11].

Lemma1. *If* $\beta > 0$ *and* $k > \gamma - \delta - 1$, *then*

$$I_{o,z}^{\beta,\gamma,\delta} z^k = \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\delta+1)} z^{k-\gamma}. \tag{9.4}$$

Employing the technique used earlier by Srivastive, Saigo and Owa [11] and with the aid of the above lemma, we can prove the following:

Theorem 13. *Let $\beta > 0, \gamma < 2, \beta + \delta > -2, \gamma - \delta < 2, \gamma(\beta + \delta) \leq 3\beta$. If the function $f(z)$ defined by* (1.8) *is in the class $T^*(A, B, n, \alpha)$, then*

$$|I_{o,z}^{\beta,\gamma,\delta} f(z)| \geq \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\delta)} \left\{1 - \frac{2(B-A)(1-\alpha)(2-\gamma+\delta)}{(2-\gamma)(2+\beta+\delta)C_2}|z|\right\} \tag{9.5}$$

and

$$|I_{o,z}^{\beta,\gamma,\delta} f(z)| \leq \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\delta)} \left\{1 + \frac{2(B-A)(1-\alpha)(2-\gamma+\delta)}{(2-\gamma)(2+\beta+\delta)C_2}|z|\right\} \tag{9.6}$$

for $z \in U$, where

$$U_o = \begin{cases} U & (\gamma \leq 1) \\ U - \{0\} & (\gamma > 1). \end{cases}$$

The equalities in (9.5) *and* (9.6) *are attained for the function $f(z)$ given by* (3.5).

Acknowledgements

The authors would like to thank the referee of the paper for his helpfull suggestions.

References

[1] M. K. Aouf, "A generalization of multivalent functions with negative coefficients," *J. Korean Math. Soc.*, 25(1988), no. 1, 53-66.

[2] R. M. Goel and N. S. Sohi, "Multivalent functions with negative coefficients," *Indian J. Pure Appl. Math.*, 12 (1981), no. 7, 844-853.

[3] V. P. Gupta and P. K. Jain, "Certain classes of univalent functions with negative coefficients," *Bull. Austral. Math. Soc.*, 14 (1976), 409-416.

[4] M. D. Hur and G. M. Oh, "On Certain class of analytic functions with negative coefficients," *Pusan Kyongnam Math. J.*, 5 (1989), 69-80.

[5] S. Owa, "On the distortion theorems," *I, Kyungpook Math. J.*, 18 (1978), 53-59.

[6] S. Owa and M. K. Aouf, "On subclasses of univalent functins with negative coefficients," *Pusan Kyongnam Math. J.*, 4 (1988), 57-73.

[7] G. S. Salagean, "Subclasses of univalent functions," *Lecture notes in Math.*(Springer-Verlag), 1013 (1983), 362-372.

[8] A. Schild and H. Silverman, "Convolutions of univalent functions with negative coefficients," *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, 29 (1975), 99-107.

[9] H. Silverman, "Univalent functions with negative coefficients," *Proc. Amer Math. Soc.*, 51 (1975), 109-116.

[10] H. M. Srivastava and S. Owa, "An application of the fractional derivative," *Math. Japon.*, 29 (1984), 383-389.

[11] H. M. Srivastava, M. Saigo and S. Owa, "A class of distortion theorems involving certain operators of fractional calculus," *J. Math. Anal. Appl.*, 131 (1988), 412-420.

Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, EGYPT.