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FUZZY IMPLICATIVE IDEALS IN BCK-ALGEBRAS

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Abstract. An implicative ideal of a BCK-algebra is defined by Iséki [1]. In 1991, Xi [7] defined a fuzzy implicative ideal of a BCK-algebra. In this paper, we investigate the properties of fuzzy implicative ideals in BCK-algebras.

The concept of fuzzy sets was introduced by Zadeh in [8]. Rosenfeld [6] applied it to the fundamental theory of groups. In [7], Xi applied the concept of fuzzy sets to BCK-algebras, and he got some interesting results. The aim of this paper is to investigate the properties of fuzzy implicative ideals in BCK-algebras.

Let us recall some definitions and results, which are necessary for development of the paper.

An algebra $(X;*,0)$ of type (2,0) is called a BCK-algebra if it satisfies the following conditions:

BCK-1 $(x*y)*(x*z) \leq z*y$,
BCK-2 $x*(x*y) \leq y$,
BCK-3 $x \leq x$,
BCK-4 $0 \leq x$,
BCK-5 $x \leq y$ and $y \leq x$ imply $x = y$,
BCK-6 $x \leq y$ if and only if $x*y = 0$,

for all $x, y, z \in X$.

In any BCK-algebra X, the following properties hold:

(1) $(x*y)*z = (x*z)*y$,
(2) $x*0 = x$,
(3) $x*y \leq x$,
(4) $(x*z)*(y*z) \leq x*y$,
(5) $x \leq y$ implies $x*z \leq y*z$ and $z*y \leq z*x$.

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A fuzzy set μ in a set X is a function from X into [0,1]. For a fuzzy set μ in X and $t \in [0, 1]$, the set

$$\mu_t = \{x \in X : \mu(x) \geq t\}$$

is called a level subset of μ.

In what follows, X would mean a BCK-algebra unless otherwise specified.

Definition 1 ([1]). A nonempty subset I of X is said to be implicative if it satisfies

(P1) $0 \in I$,

(P2) $(x * y) * z \in I$ and $y * z \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

Definition 2 ([7]). A fuzzy set μ in X is called a fuzzy implicative ideal (briefly, a *f.i.* ideal) of X if

(F1) $\mu(0) \geq \mu(x)$ for all $x \in X$,

(F2) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}$ for all $x, y, z \in X$.

Example 1. Let I be an implicative ideal of X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I, \\ t & \text{if } x \in I, \end{cases}$$

where t is a fixed number in (0,1). Then μ is a *f.i.* ideal of X. In fact, it is clear that $\mu(0) \geq \mu(x)$ for all $x \in X$. In order to prove that μ satisfies (F2), we will divide into the following cases:

If $(x*y)*z \in I$ and $y*z \in I$, then $x*z \in I$. Thus $\mu(x*z) = \mu((x*y)*z) = \mu(y*z) = t$, and so

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

If $(x * y) * z \notin I$ and $y * z \notin I$, then $\mu((x * y) * z) = \mu(y * z) = 0$. Hence

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

If exactly one of $(x * y) * z$ and $y * z$ belongs to I, then exactly one of $\mu((x * y) * z)$ and $\mu(y * z)$ is equal to 0. Thus

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

Consequently μ is a *f.i.* ideal of X.

Theorem 1 ([7]). *Let μ be a fuzzy set in X. Then μ is a f.i. ideal of X if and only if μ_t is an implicative ideal of X for all $t \in [0.1]$, when $\mu_t \neq \varnothing$.*

Definition 3. Let μ be a *f.i.* ideal of X. The implicative ideals μ_t, $t \in [0, 1]$, are called level implicative ideals of X.

Note that if X is a finite BCK-algebra, then the number of implicative ideals of X is finite whereas the number of level implicative ideals of a *f.i.* ideal μ appears to be infinite. But, since every level implicative ideal is indeed an implicative ideal of X, not all these level implicative ideals are distinct. The next theorem characterizes this aspect.

Theorem 2. *If μ is a f.i. ideal of X, then two level implicative ideals μ_{t_1} and μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in X$ such that $t_1 \leq \mu(x) < t_2$.*

Proof. Assume that $\mu_{t_1} = \mu_{t_2}$ for $t_1 < t_2$ and that there exists $x \in X$ such that $t_1 \leq \mu(x) < t_2$. Then μ_{t_2} is a proper subset of μ_{t_1}, which is impossible.

Conversely suppose that there is no $x \in X$ such that $t_1 \leq \mu(x) < t_2$. Note that $t_1 < t_2$ implies $\mu_{t_2} \subseteq \mu_{t_1}$. If $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$. Since $\mu(x) \nless t_2$, it follows that $\mu(x) \geq t_2$, so that $x \in \mu_{t_2}$. This shows that $\mu_{t_1} = \mu_{t_2}$. This completes the proof.

Remark 1. As a consequence of Theorem 2, the level implicative ideals of a *f.i.* ideal μ of a finite BCK-algebra X form a chain. But $\mu(0) \geq \mu(x)$ for all $x \in X$. Therefore μ_{t_0}, where $t_0 = \mu(0)$, is the smallest level implicative ideal but not always $\mu_{t_0} = \{0\}$ as shown in the following example, and hence we have the chain:

$$\mu_{t_0} \subset \mu_{t_1} \subset \cdots \subset \mu_{t_r} = X$$

where $t_0 > t_1 > \ldots > t_r$.

Notation. $Im(\mu)$ denotes the image set of μ.

Example 2. Let I be a nonzero implicative ideal of X and let μ be the *f.i.* ideal of X as in Example 1. Then $Im(\mu) = \{0, t\}$. Further, the two level implicative ideals of μ are $\mu_0 = X$ and $\mu_t = I$. Thus we have that $\mu(0) = t$ but $\mu_t = I \neq \{0\}$.

Theorem 3. *Let μ be a f.i. ideal of X. If $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$, where $t_1 < t_2 < \ldots < t_n$, then the family of implicative ideals μ_{i_i} $(i = 1, 2, \cdots, n)$ constitutes all the level implicative ideals of μ.*

Proof. Let $t \in [0,1]$ and $t \notin Im(\mu)$. If $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = X$, therefore $\mu_t = X$ and $\mu_t = \mu_{t_1}$. If $t_i < t < t_{i+1}$ $(1 \leq i \leq n-1)$, then there is no $x \in X$ such that $t \leq \mu(x) < t_{i+1}$. It follows from Theorem 2 that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in [0,1]$, the level implicative ideal μ_t is in $\{\mu_{t_i} : i = 1, 2, \ldots, n\}$.

Lemma 1. *Let μ be a f.i. ideal of a finite BCK-algebra X. If s and t belong to $Im(\mu)$ such that $\mu_s = \mu_t$, then $s = t$.*

Proof. Assume that $s \neq t$, say $s < t$. Then there is $x \in X$ such that $\mu(x) = s < t$, and so $x \in \mu_s$ and $x \notin \mu_t$. Thus $\mu_s \neq \mu_t$, a contradiction. The proof is complete.

Theorem 4. *Let μ and ν be two f.i. ideals of a finite BCK-algebra X with identical family of level implicative ideals. If $Im(\mu) = \{t_1, t_2, \cdots, t_m\}$ and $Im(\nu) = \{s_1, s_2, \ldots, s_n\}$, where $t_1 > t_2 > \ldots > t_m$ and $s_1 > s_2 > \cdots > s_n$, then*

(a) $m = n$;

(b) $\mu_{t_i} = \nu_{s_i}, i = 1, \ldots, m$;

(c) *if $x \in X$ such that $\mu(x) = t_i$ then $\nu(x) = s_i, i = 1, \ldots, m$.*

Proof. By means of Theorem 3, we know that the only level implicative ideals of μ and ν are μ_{t_i} and ν_{s_i}, respectively. Since μ and ν have the identical family of level implicative ideals, it follows that $m = n$. Thus (a) holds. Using Theorem 3 again, we get that $\{\mu_{t_1}, \ldots, \mu_{t_m}\} = \{\nu_{s_1}, \ldots, \nu_{s_n}\}$, and by Theorem 2 we have

$$\mu_{t_1} \subset \mu_{t_2} \subset \cdots \subset \mu_{t_m} = X \quad \text{and} \quad \nu_{s_1} \subset \nu_{s_2} \subset \cdots \subset \nu_{s_n} = X$$

Hence $\mu_{t_i} = \nu_{s_i}$, $i = 1, \cdots, m$ and (b) holds.

Let $x \in X$ be such that $\mu(x) = t_i$ and let $\nu(x) = s_j$. Noticing that $x \in \nu_{s_i}$, that is, $\nu(x) \geq s_i$, we obtain $s_j \geq s_i$. Thus $\nu_{s_j} \subseteq \nu_{s_i}$. Since $x \in \nu_{s_j}$ and $\nu_{s_j} = \mu_{t_j}$, therefore $x \in \mu_{t_j}$ and so $t_i = \mu(x) \geq t_j$. It follows that $\mu_{t_i} \subseteq \mu_{t_j}$. By (b), $\nu_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \nu_{s_j}$. Consequently $\nu_{s_i} = \nu_{s_j}$, and by Lemma 1 we conclude that $s_i = s_j$. Thus $\nu(x) = s_i$. The proof is complete.

Theorem 5. *Let μ and ν be two f.i. ideals of a finite BCK-algebra X such that the families of level implicative ideals of μ and ν are identical. Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Assume that $Im(\mu) = Im(\nu) = \{t_1, \ldots, t_n\}$ where $t_1 > \cdots > t_n$. Let $x_1, \ldots, x_n$ be distinct elements of X such that $\mu(x_i) = t_i (1 \leq i \leq n)$. By Theorem 4(c), $\nu(x_i) = t_i$ $(1 \leq i \leq n)$. Since for any $x \in X$ there exists some t_i such that $\mu(x) = t_i$, therefore $x \in \mu_{t_i}$. Hence $\nu(x) \geq t_i = \mu(x)$. By the same argument, we obtain $\mu(x) \geq \nu(x)$. Consequently $\mu(x) = \nu(x)$ for all $x \in X$. This completes the proof.

Theorem 6. *Let X be a finite BCK-algebra and let μ be a fuzzy set in X with $Im(\mu) = \{t_o, t_1, \cdots, t_k\}$ where $t_o > t_1 > \ldots > t_k$. If there exists a chain of implicative ideals of X:*

$$I_0 \subset I_1 \subset \cdots \subset I_k = X$$

such that $\mu(\overline{I}_n) = t_n$, where $\overline{I}_n = I_n - I_{n-1}$, $I_{-1} = \varnothing, n = 0, 1, \ldots, k$, then μ is a f.i. ideal of X.

Proof. Since $0 \in I_0$, we have $\mu(0) = t_0 \geq \mu(x)$ for all $x \in X$. We divide into the following cases to prove that μ satisfies (F2): If $(x * y) * z \in \overline{I}_n$ and $y * z \in \overline{I}_n$, then $x * z \in I_n$ because I_n is an implicative ideal of X . Thus

$$\mu(x * z) \geq t_n = \min\{\mu((x * y) * z), \mu(y * z)\}$$

If $(x * y) * z \notin \overline{I}_n$ and $y * z \notin \overline{I}_n$, then the following four cases arise:

1. $(x * y) * z \in X - I_n$ and $y * z \in X - I_n$,
2. $(x * y) * z \in I_{n-1}$ and $y * z \in I_{n-1}$,
3. $(x * y) * z \in X - I_n$ and $y * z \in I_{n-1}$,
4. $(x * y) * z \in I_{n-1}$ and $y * z \in X - I_n$.

But, in either case, we know that

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

If $(x * y) * z \in \overline{I}_n$ and $y * z \notin \overline{I}_n$, then either $y * z \in I_{n-1}$ or $y * z \in X - I_n$. It follows that either $x * z \in I_n$ or $x * z \in X - I_n$. Thus

$$\mu(x * z) \geq t_n = \min\{\mu((x * y) * z), \mu(y * z)\}.$$

If $(x * y) * z \notin \overline{I}_n$ and $y * z \in \overline{I}_n$, then by similar process we have

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

Summarizing the above results, we obtain

$$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y * z)\}.$$

for all $x, y, z \in X$. Consequently μ satisfies the condition (F2). This complietes the proof.

Theorem 7. *Let μ be a f.i. ideal of a finite BCK-algegra X. If $Im(\mu) = \{t_o, t_1, \ldots, t_k\}$ where $t_o > t_1 > \cdots > t_k$, then*
(a) $I_n = \mu_{t_n} (n = 0, 1, \ldots, k)$ is an implicative ideal of X,
(b) $\mu(\overline{I}_n) = t_n (n = 0, 1, \ldots, k)$ where $\overline{I}_n = I_n - I_{n-1}$ and $I_{-1} = \varnothing$,

Proof. (a) is by Theorem 3.

(b) Obviously $\mu(I_0) = t_0$. Since $\mu(I_1) = \{t_0, t_1\}$, for $x \in \overline{I}_1$ we have $\mu(x) = t_1$. Hence $\mu(\overline{I}_1) = t_1$. Repeating the above argument, we have $\mu(\overline{I}_n) = t_n (n = 0, 1, \ldots, k)$. This completes the proof.

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References

[1] K. Iséki, "On ideals in BCK-algebras," *Math. Seminar Notes* (Presently, Kobe J. Math), 3 (1975), 1-12.

[2] K. Iséki and S. Tanaka, " Ideal theory of BCK-algebras," *Math. Japon.*, 21 (1976), 351-366.

[3] Y. B. Jun, "Characterizations of fuzzy ideals by their level ideals in BCK(BCI)-algebras," *Math. Japon.*, 38 (1993), 67-71.

[4] Y. B. Jun, S. M. Hong, J. Meng and X. L. Xin, "Characterizations of fuzzy positive implicative ideals in BCK-algebras," *Math. Japon.*, 40 (1994), 503-507.

[5] J. Meng, "Ideals in BCK-algebras (Chinese)," *Pure and Appl. Math.*, 2 (1986), 68-76.

[6] A. Rosenfeld, "Fuzzy groups," *J. Math. Anal. Appl.*, 35 (1971), 512-517.

[7] O. G. Xi, "Fuzzy BCK-algebras," *Math. Japon.*, 36 (1991), 935-942.

[8] L. A. Zadeh, "Fuzzy sets," *Inform. Control,* 8 (1965), 338-353.

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