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THE GAP OF THE GRAPH OF A LINEAR TRANSFORMATION

JAVAD FAGHIH-HABIBI AND RALPH HOLLINGSWORTH

Abstract. Assume that A is a linear transformation from C^n into C^m. The Gap of the graph of A is

$$\theta(A_g)=\frac{||A||}{\sqrt{1+||A||^2}}.$$

Here $||A||$ is the operator norm of A.

This is an extention of the result in [2], in which the first author used another method to prove for the case of m=n.

1. Introduction

Consider C^k with the usual scalar product and the corresponding norm. Let S_i be the unit sphere in the subspace $M_i \subseteq C^k$ for $i=1,2$.

The gap between M_1 and M_2 is defined as

$$\theta(M_1,M_2)=\max\left\{\sup_{x\in S_1} d(x,M_2),\ \sup_{x\in S_2} d(x,M_2)\right\}$$

where $d(x,M)=\inf_{t\in M}||x-t||$ is the distance of x from the set $M\subseteq C^k$.

The gap is being used to determine stable invariant subspaces of a linear transformation which is important from the point of view of numerical computations.

See Gohberg, Lancaster and Rodman [1].

Let $A\ :\ C^n\longrightarrow C^m$ be a linear transformation. We assume that A is given by an $m\times n$ matrix with respect to the standard orthonormal bases. Let

$$G_A=\left\{\begin{pmatrix} x\\ Ax\end{pmatrix} : x\in C^n\right\}\subset C^n\oplus C^m$$

be the graph of A. The norm of A is

$$||A|| = \sup_{||x||=1} ||Ax||.$$

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The case $m = n$ has been dealt with in [2] with different proof.

2. The Gap of the Graph of A

The gap of the graph of A is the gap between two subspaces G_A and $G_0, \theta(G_A, G_0)$, which we denote by $\theta(A_g)$. Set

$$f(x,y) = d(\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ Ay \end{pmatrix}) = \sqrt{\|x-y\|^2 + \|Ay\|^2}.$$

Then

$$\theta(A_g) = \max\left\{ \sup_{\|x\|=1} [\inf_y f(x,y)], \sup_{\|y\|^2+\|Ay\|^2=1} [\inf_x f(x,y)] \right\}.$$

Note that

$$f(x,y) \geq \sqrt{\|x\|^2 + \|y\|^2 + \|Ay\|^2 - 2\|x\| \quad \|y\|}$$

and the equality sign holds if $y = cx$ for some $c > 0$.
Choose $y = cx$ for some $c > 0$, we have (assuming $\|x\| = 1$)

$$f(x,y) = \sqrt{(1 + \|Ax\|^2)c^2 - 2c + 1}.$$

Set $c = \frac{1}{1+\|Ax\|^2}$, to obtain

$$\inf_y f(x,y) = \frac{\|Ax\|}{\sqrt{1 + \|Ax\|^2}}.$$

Therefore

$$\sup_{\|x\|=1} [\inf_y f(x,y)] = \sup_{\|x\|=1} \frac{\|Ax\|}{\sqrt{1 + \|Ax\|^2}}.$$

Since $\frac{t}{\sqrt{1+t^2}}$ is an increasing function for $t \geq 0$, and $\|A\| = \sup_{\|x\|=1} \|Ax\|$, we have

$$\sup_{\|x\|=1} [\inf_y f(x,y)] = \frac{\|A\|}{\sqrt{1 + \|A\|^2}}.$$

Now consider $y \in C^n$ with $\|y\|^2 + \|Ay\|^2 = 1$. We see that

$$f(x,y) \geq \sqrt{\|x\|^2 - 2\|x\| \, \|y\| + 1}.$$

Set $x = cy$ for some $c > 0$ to get

$$f(x,y) = \sqrt{\|y\|^2 c^2 - 2c\|y\|^2 + 1}.$$

Choosing $c = 1$, we have

$$\inf_x f(x,y) = \sqrt{1 - \|y\|^2} = \|Ay\|.$$

Setting $y = kz$, with $\|z\| = 1$, we have $\sup\{\|Ay\| : \|y\|^2 + \|Ay\|^2 = 1\} = \sup\{| k | \|Az\| :| k |^2 (1 + \|Az\|^2) = 1\} = \sup_{\|z\|=1} \frac{\|Az\|}{\sqrt{1+\|Az\|^2}}$.

Sicne $\sup_{\|z\|=1} \|Az\| = \|A\|$, and $\frac{t}{\sqrt{1+t^2}}$ is an increasing continuous function of $t \geq 0$, we have

$$\sup \frac{\|Az\|}{\sqrt{1 + \|Az\|^2}} = \frac{\|A\|}{\sqrt{1 + \|A\|^2}}, \text{ and}$$

$$\sup_{\|y\|^2+\|Ay\|^2=1} [\inf_x f(x,y)] = \frac{\|A\|}{\sqrt{1 + \|A\|^2}}.$$

Therefore

$$\theta(A_g) = \frac{\|A\|}{\sqrt{1 + \|A\|^2}}.$$

References

[1] I. Gohberg, P. Lancaster and L. Rodman., *Invariant Subspaces of Matrices with Applications,* John Wiley and Sons, New York, 1986.

[2] J. F. Habibi, "The gap of the graph of a matrix," *Linear Algebra Appl.,* 186 (1993), 55-57.

Department of Mathematics and Computer Science, Muskingum College, New Concord, Ohio 43762, U.S.A.