TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 2, Summer 1995

λ-CONVERGENCE AND λ-REPLACEABILITY

WOLFGANG BEEKMANN AND SHAO-CHIEN CHANG

Abstract. E. Jürimäe has introduced the notion of λ-replaceability. We shall show that this notion can be characterized in ordinary summability terms.

Let A be an infinite matrix with complex entries, w be the set of all complex sequences and c be the set of all convergent sequences. We let

$$c_A := \{x \in w \mid Ax \in c\}$$

-the summability domain for A. It is known that c_A is an FK-space.

The term replaceable matrix was first introduced in the 60s by A. Wilansky and further studied by many including the authors. Noticeably, distinguished subsets were frequently used in these studies.

In the early 60s, the notion of λ-convergence was introduced by the Tartu School of Mathematical Analysis. For a given positive sequence $\{\lambda_n\} \nearrow \infty$, we define the set

$$c^\lambda := \{x \in c \mid \lim_{n\to\infty} \lambda_n(x_n - \lim x) \text{ exists } \}.$$

Those sequences in c^λ are the λ-convergent sequences or the sequences convergent with speed λ.

For a given matrix A, the λ-summability domain of A, c_A^λ, is defined by

$$c_A^\lambda := \{x \in c_A \mid Ax \in c^\lambda\}.$$

We have proved [2] that $\exists$ matrices D and E such that $c_D = c^\lambda$ and $c_E = c_A^\lambda$. Hence one can now express those distinguished subsets of c_A^λ in terms of ordinary matrix summability. Also, replaceability of the ordinary summability domain $c_E = c_A^\lambda$ is defined

Received January 7, 1994; revised May, 3, 1994.

1991 *Mathematics Subject Classification.* 40C05, 40C99.

Key words and phrases. λ-convergence and λ-replaceability, matrix summability, replaceable matrices.

as replaceability of the matrix E (see e.g. [1]) which means that E can be replaced by a matrix with column limits zero without changing the summability domain.

We shall assume that all λ-summability domains like c_A^λ contain φ - the set of all finite sequences throughout in our discussion.

It is known (see Jürimäe) that for given $f \in (c_A^\lambda)'$, we have

$$f(x) = \mu_1 \lim_A x + \mu_2 \lim_n \gamma_n(x) + \sum_k t_k \gamma_k(x) + sx \text{ for } x \in c_A^\lambda, \tag{1}$$

where $\mu_1, \mu_2 \in \mathbb{C}$, $\gamma_n(x) = \lambda_n(\sum_k a_{nk}x_k - \lim_A x)$, $s \in (c_A^\lambda)^\beta = \{x \mid sx = \sum_k s_k x_k$ converges $\}$ and $t = (t_k) \in \ell$.

The term λ-replaceable matrix was introduced by E. Jürimäe. We shall use the following characterization for λ-replaceable matrices due to Jürimäe [3], Th. 3.2:

Theorem 1. *An infinite matrix D is λ-replaceable if and only if $\exists$ a matrix A such that $c_A^\lambda = c_D^\lambda$ and $f|_\varphi = 0$ for some $f \in (c_A^\lambda)'$ with $\mu_2 \neq 0$ in the representation* (1).

We are now in a position to prove the following:

Theorem 2. *D is λ-replaceable, if and only if c_D^λ is replaceable when considered as an ordinary summability domain.*

Proof. If D is λ-replaceable, then we can choose A and $f \in (c_A^\lambda)'$ such that $f|\varphi = 0$ and $\mu_2 \neq 0$ in (1). Let E be a matrix such that $c_A^\lambda = c_E$. The proof of theorem 4 in [2] shows that E can be chosen such that $\lim_n \gamma_n(x) = \lim_E x$ for all $x \in c_D^\lambda$.

With some $\mu = \mu_E(f) \in \mathbb{C}$, $\tilde{t} \in \ell$, $\tilde{s} \in c_E^\beta$, we have a standard representation

$$f(x) = \mu \lim_E x + \tilde{t}(Ex) + \tilde{s}x \text{ for } x \in c_E. \tag{2}$$

But then $\mu = \mu_2 \neq 0$ (see [2]). From a result of Zeller, cf. [4], there is a matrix Y such that $c_Y = c_E$ and $f = \lim_Y$, hence $\lim_Y |_\varphi = 0$. This says that E is replaceable, i.e. $c_E = c_D^\lambda$ is replaceable.

Conversely, let E be a matrix such that $c_D^\lambda = c_E$ and E be replaceable. We now let Y be a matrix with $c_Y = c_E$ and $\lim_Y |_\varphi = 0$; in here $\mu_Y(\lim_Y) = 1$, hence, for $f := \lim_Y$, we have a representation (2) of f with $\mu \neq 0$ (see [1]). But then, representing f in the form (1), $\mu_2 = \mu \neq 0$, see [2]. Hence, by [3], Th. 2.3, there exists a matrix A with $c_A^\lambda = c_D^\lambda$ and $\lim_n \gamma_n(x) = f(x)$ for $x \in c_D^\lambda$. The λ-replaceability of D now follows.

References

[1] W. Beekmann and S.-C. Chang, "Replaceability and μ-uniqeness - a unified approach," *C. R. Math. Rep. Sci. Canada*, 6 (1984), 113-116.

[2] W. Beekmann and S.-C. Chang, "λ-convergence and λ-conullity," *Zeit. für Anal. Anw.* 12 (1993), 179-182.
[3] E. Jürimäe, "Replaceability and λ-summability," *Israel Math. Conf. Proc.* 4 (1991), 157-162.
[4] K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren,* 2. Aufl., Berlin-Heidelberg-New York, Springer-Verlag, 1970.

Fachbereich Mathematik, Fern Universität - Gesamthochschule, 58084 Hagen, Germany.

Department of Mathematics, Brock University, St. Catharines, Ontario, Canada L2S 3A1.