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ON NONLINEAR INTEGRODIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. We prove an existence theorem for nonlinear integrodifferential equations with deviating arguments using the Schauder fixed theorem.

1. Introduction

Several authors have studied the problem of existence of solutions of nonlinear integrodifferential equations with deviating arguments [5, 7, 8]. Banaś and Stopka [3] have proved an existence theorem for a differential equation with deviating argument using the measure of noncompactness. Balachandran and Ilamaran [1, 2] have proved existence theorems for nonlinear integral equations with deviating arguments. In this paper we shall derive a set of sufficient conditions for the existence of a solution of integrodifferential equatons with deviatig arguments. The technique used in this paper is similar to that used by Banaś and Stopka [3] and Balachandran and Ilamaran [1, 2].

2. Basic Assumptions

Let $p(t)$ be a given continuous function defined on the interval $[0,\infty)$ and taking real positive values. Denote by $C_p = C([0,\infty), p(t) : R)$ the set of all continuous functions from $[0,\infty)$ into R such that

$$\sup\{|x(t)|p(t) : t \geq 0\} < \infty.$$

It has been proved [6] that C_p forms a real Banach space with regard to the norm

$$\| x \| = \sup\{|x(t)|p(t) : t \geq 0\}.$$

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If $x \in C_p$ then we will denote $w^T(x,\varepsilon)$ the usual modulus of continuity of x on the interval $[0,T]$ i.e.,

$$w^T(x,\varepsilon) = \sup\{|x(t)-x(s)| : |t-s| \le \varepsilon, t, s \in [0,T]\}.$$

Our existence theorem is based on the following lemma.

Lemma [4] *Let E be a bounded set in the space C_p. If all functions belonging to E are equicontinuous on each interval $[0,T]$ and $\lim_{T\to\infty} \sup\{|x(t)|p(t) : t \ge T\} = 0$ uniformly with respect to E, then E is relatively compact in C_p.*

Consider the integrodifferential equation of the form

$$x'(t) = f(t, x(H(t)), \int_0^t K(t,s,x'(h(s)))ds) \tag{1}$$

with the initial condition

$$x(0) = 0. \tag{2}$$

If we define $x'(t) = y(t)$ then the equation (1) with condition (2) will be transformed into the following functional-integral equation

$$y(t) = f(t, \int_0^{H(t)} y(s)ds, \int_0^t K(t,s,y(h(s)))ds). \tag{3}$$

Assume the following conditions:

(i). The function $f : [0,\infty) \times R^2 \to R$ is continuous and there exist positive constants A_1 and A_2 such that

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le A_1|x_1 - x_2| + A_2|y_1 - y_2|.$$

(ii). Let $\Delta = \{(t,s) : 0 \le s \le t < \infty\}$.

The function $K : \Delta \times R \to R$ is continuous and there exist continuous functions $m : \Delta \to [0,\infty), a : [0,\infty) \to [1,\infty), b : [0,\infty) \to [1,\infty)$ such that

$$|K(t,s,y)| \le m(t,s) + a(t)b(s)|y|$$

for all $(t,s) \in \Delta$ and $y \in R$.

In order to formulate other assumptions let us define

$$L(t) = \int_0^t a(s)b(s)ds, \quad t \ge 0.$$

Take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)e^{ML(t)+t}]^{-1}$.

(iii). The function $H : [0,\infty) \to [0,\infty)$ is continuous and $H(t) \geq t$ for all $t \in [0,\infty)$ and the number

$$m = \sup\{e^{-ML(t)} \int_t^{H(t)} a(s)e^{ML(s)}ds : t \geq 0\} < \infty.$$

(iv). The function $h : [0,\infty) \to [0,\infty)$ is continuous and $h(t) \leq t$ for all $t \in [0,\infty)$ and there exists a constant $Q > 0$ such that

$$a(h(t)) \leq Qa(t).$$

(v). There exist constants $B_1 > 0, B_2 > 0$ such that for any $t \in [0,\infty)$ the following inequalities holds

$$\int_0^t m(t,s)ds \leq B_1 a(t)e^{ML(t)}$$

and

$$|f(t,0,0)| \leq B_2 a(t)e^{ML(t)}.$$

(vi). $[A_1\{(1/M) + m\} + A_2\{B_1 + (Q/M)\} + B_2] < 1.$

3. Existence Theorem

Theorem: *Assume that the hypotheses (i) to (vi) hold. Then the equatoin (2) has at least one solution y in the space C_p such that $|y(t)| \leq a(t)e^{ML(t)}$ for any $t \geq 0$.*

Proof. Define a transformation F in the space C_p by

$$(Fy)(t) = f(t, \int_0^{H(t)} y(s)ds, \int_0^t K(t,s,y(h(s)))ds).$$

From our assumptions we observe that $(Fy)(t)$ is continuous on the interval $[0,\infty)$. Define the set E in C_p by

$$E = \{y \in C_p : |y(t)| \leq a(t)e^{ML(t)}\}.$$

Clearly E is nonempty, bounded, convex and closed in C_p . Now we prove that F maps the set E into itself. Take $y \in E$. Then from our assumptions we have

$$\begin{aligned}
|(Fy)(t)| \leq & A_1 \int_0^{H(t)} |y(s)|ds + A_2 \int_0^t |K(t,s,y(h(s)))|ds + |f(t,0,0)| \\
\leq & A_1 \int_0^t a(s)e^{ML(s)}ds + A_1 \int_t^{H(t)} a(s)e^{ML(s)}ds + A_2 \int_0^t m(t,s)ds \\
& + A_2 \int_0^t a(t)b(s)|y(h(s))|ds + B_2 a(t)e^{ML(t)} \\
\leq & (A_1/M) \int_0^t Mb(s)a(s)e^{ML(s)}ds + A_1 m e^{ML(t)} + A_2 B_1 a(t)e^{ML(t)} \\
& + A_2 a(t) \int_0^t b(s)a(h(s))e^{ML(h(s))}ds + B_2 a(t)e^{ML(t)} \\
\leq & (A_1/M)a(t)e^{ML(t)} + A_1 m a(t)e^{ML(t)} + A_2 B_1 a(t)e^{ML(t)} \\
& + (A_2 Q/M)a(t) \int_0^t Mb(s)a(s)e^{ML(s)}ds + B_2 a(t)e^{ML(t)} \\
\leq & [A_1\{(1/M)+m\} + A_2\{B_1 + (Q/M)\} + B_2]a(t)e^{ML(t)} \\
\leq & a(t)e^{ML(t)}
\end{aligned}$$

which proves that $FE \subset E$.

Now we want to prove that F is continuous on the set E. For this let us fix $\varepsilon > 0$ and take $y, z \in E$ such that $\| y - z \| \leq \varepsilon$. Further take an arbitrary fixed $T > 0$. In view of (ii) the function $K(t,s,y)$ is uniformly continuous on

$$[0,T] \times [0,T] \times [-r(h(T)), r(h(T))]$$

where $r(h(T)) = \max\{a(h(s))e^{ML(h(s))} : s \in [0,T]\}$. Thus, we have for $t \in [0,T]$

$$\begin{aligned}
& |(Fy)(t) - (Fz)(t)| \\
= & \left| f(t, \int_0^{H(t)} y(s)ds, \int_0^t K(t,s,y(h(s)))ds) - f(t, \int_0^{H(t)} z(s)ds, \int_0^t K(t,s,z(h(s)))ds) \right| \\
\leq & A_1 \int_0^{H(t)} |y(s) - z(s)|ds + A_2 \int_0^t |K(t,s,y(h(s))) - K(t,s,z(h(s)))|ds \\
\leq & A_1 A H(t) e^{ML(H(t))+H(t)} \| y - z \| + A_2 \int_0^t |K(t,s,y(h(s))) - K(t,s,z(h(s)))|ds \\
\leq & A_1 A B e^{ML(B)+B} \varepsilon + \beta(\varepsilon) \qquad (4)
\end{aligned}$$

where $B = \max\{H(t) : t \in [0,T]\}$, $A = \max\{a(t) : t \in [0,B]\}$ and β is some continuous function having the property $\lim_{\varepsilon \to 0} \beta(\varepsilon) = 0$. Further, let us take $t \geq T$. Then we have

$$\begin{aligned}
|(Fy)(t) - (Fz)(t)| \leq & |(Fy)(t)| + |(Fz)(t)| \\
\leq & 2a(t)e^{ML(t)}
\end{aligned}$$

$$|(Fy)(t)-(Fz)(t)|p(t)\le 2e^{-t}.$$

Hence for T sufficiently large we have

$$|(Fy)(t)-(Fz)(t)|p(t)\le \varepsilon. \tag{5}$$

By (4) and (5) we get that F is continuous on the set E.

Now we prove that FE is relatively compact. For every $y\in E$ we have $Fy\in E$ which gives $|(Fy)(t)|p(t)\le e^{-t}$. Hence

$$\lim_{T\to\infty}\sup\{|(Fy)(t)|p(t): t\ge T\}=0$$

uniformly with respect to $y\in E$.

Furthermore, let us fix $\varepsilon>0, T>0$ and $t,s\in[0,T]$ such that $|t-s|\le\varepsilon$ and $s\le t$. Then for $y\in E$, we have

$$\begin{aligned}
&|(Fy)(t)-(Fy)(s)|\\
\le& f(t,\int_0^{H(t)}y(u)du,\int_0^t K(t,u,y(h(u)))du)\\
&-f(t,\int_0^{H(s)}y(u)du,\int_0^s K(s,u,y(h(u)))du)|\\
&+|f(t,\int_0^{H(s)}y(u)du,\int_0^s K(s,u,y(h(u)))du)\\
&-f(s,\int_0^{H(s)}y(u)du,\int_0^s K(s,u,y(h(u)))du)|\\
\le& A_1\int_{H(s)}^{H(t)}|y(u)|du+A_2|\int_0^t K(t,u,y(h(u)))du-\int_0^s K(s,u,y(h(u)))du|\\
&+w^T(f,\varepsilon).\\
\le& A_1Ae^{ML(B)}w^T(H,\varepsilon)+A_2|\int_0^t K(t,u,y(h(u)))du-\int_0^s K(t,u,y(h(u)))du|\\
&+A_2|\int_0^s K(t,u,y(h(u)))du-\int_0^s K(s,u,y)ch(u)))du|+w^T(f,\varepsilon)\\
\le& A_1Ae^{ML(B)}w^T(H,\varepsilon)+A_2\int_s^t|K(t,u,y(h(u)))|du\\
&+A_2\int_0^s|K(t,u,y(h(u)))-K(s,u,y(h(u)))|du+w^T(f,\varepsilon)\\
\le& A_1Ae^{ML(B)}w^T(H,\varepsilon)+A_2\varepsilon\max\{m(t,u)+a(t)b(u)[p(h(u))]^{-1}: 0\le u\le t\le T\}\\
&+A_2Tw^T(K,\varepsilon)+w^T(f,\varepsilon)
\end{aligned}$$

which tends to zero as $\varepsilon\to 0$. Thus FE is equicontinuous on $[0,T]$.

Therefore from the lemma FE is relatively compact. Thus the Schauder fixed point theorem guarantees that F has a fixed point $y\in E$ such that $(Fy)(t)=y(t)$. Hence the theorem is proved.

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