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A NOTE ON A GENERAL CLASS OF ARITHMETIC MEANS

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Abstract. The object of this note is to introduce a general class of arithmetic means for summing divergent series. A q-analogue is also presented.

Let α and β be two real numbers with $\alpha > 0$ and $\beta > -1$. Linear relations between two sequences $\{s_n\}$ and $\{t_n\}$ of the form

$$t_m = \sum_{k=0}^{mq} s_k \frac{(mq)!(k+\alpha+k\beta)}{(mq-k)!(mq+\alpha)(mq+\alpha+\beta)\cdots(mq+\alpha+k\beta)} \tag{1}$$

where $m = 0, 1, 2, \ldots$, are called relations of the type $P(q, \alpha, \beta)$. It is known that the familiar summability methods due to Vallée-Poussin, Obreshkov, Cesaro and Euler respectively are all of the type $P(q, \alpha, \beta)$ for particular values of the parameters. That the transformation (1) carries the identity sequence $\{1\}$ into itself was investigated earlier by Egorychev [1,2] in his research on combinatorial sums.

In what follows we will consider an extension of (1). Let $\{\lambda_k\}$ be a sequence of real numbers with $\lambda_{k+1} > -k$ $(k = 0, 1, 2, \ldots)$ such that the following sequence of polynomials

$$\phi(x, k) = \prod_{i=1}^{k} (x + \lambda_i) \tag{2}$$

differ from zero for integer $x \geq 0$, with $\phi(x, 0) = 1$. Also we denote $[x]_k = x(x-1)\cdots(x-k+1)$, the falling k factorial with $[x]_0 = 1$. Then in contrast with (1) we may introduce a wider class of linear transformations of the form

$$t_n = \sum_{k=0}^{n} s_k \frac{[n]_k (k + \lambda_{k+1})}{\phi(n, k+1)}, \tag{3}$$

where $n = 0, 1, 2, \ldots$. Clearly (1) is included in (3) with $n :=$ mq and $\lambda_i := \alpha + (i-1)\beta$. Indeed, (3) offers an extensive class of arithmetic means containing $\{\lambda_k\}$ as a sequence of free parameters.

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Theorem 1. *The linear relations defined by* (3) *yield a class of arithmetic means which are regular, namely* $t_n \to s \quad (n \to \infty)$ *whenever* $s_k \to s \quad (k \to \infty)$.

Proof. In the first place, it is plain that

$$c_{nk} := \frac{[n]_k(k+\lambda_{k+1})}{\phi(n,k+1)} > 0 \tag{4}$$

holds for all $k \leq n (n = 1, 2, \ldots)$, as $k + \lambda_{k+1} > 0$ and $\phi(n, k+1) > 0 \quad (k \leq n)$. Moreover, it is easy to verify the identity

$$\sum_{k=0}^{n} c_{nk} = 1. \tag{5}$$

Actually, (5) can be obtained by the summand-splitting

$$c_{nk} = \frac{[n]_k}{\phi(n,k)} - \frac{[n]_{k+1}}{\phi(n,k+1)}$$

and the diagonal-cancelling, noting that $[n]_0 = \phi(n, 0) = 1$ and $[n]_{n+1} = 0$. Finally, since $[n]_k$ and $\phi(n, k+1)$ contained in the left-hand side of (4) are polynomials in n of degrees k and $k+1$, respectively, it follows that

$$\lim_{n\to\infty} c_{nk} = 0 \tag{6}$$

is true for each fixed $k \geq 0$. Thus, as a consequence of the classic Toeplitz theorem [3] we see that the transformation given by (3) is regular. This completes the proof.

We are now going to construct a q-analogue of (3). For fixed $q \neq 1$, denote by $(x)_k$ the q-rising factorial

$$(x)_k = (1-x)(1-xq)\cdots(1-xq^{k-1}), \quad (x)_0 = 1.$$

Let us define

$$\phi(x, k : q) = \prod_{i=1}^{k} (\mu_i - q^x) \tag{7}$$

with $\phi(x, 0 : q) = 1$, where $\{\mu_k\}$ is any given sequence of real numbers satisfying the condition

$$\mu_{k+1} < q^k \quad (k = 0, 1, 2, \ldots). \tag{8}$$

Moreover, in contrast with (4) let us denote

$$c_{nk}^* := \frac{(q^{n-k+1})_k(\mu_{k+1} - q^k)}{\phi(n, k+1 : q)} q^{\binom{k}{2}}. \tag{9}$$

Lemma. *Let $q > 1$ and the condition* (8) *be satisfied. Then the following assertions are valid:*

(i) $c^*_{nk} > 0$ *for* $k \leq n$.

(ii) $\sum_{k=0}^{n} c^*_{nk} = 1$ *for* $n = 0, 1, 2, \ldots$.

(iii) $\lim_{n\to\infty} c^*_{nk} = 0$ *for each fixed* $k \geq 0$.

Proof. (i) is obvious as the numerator and denominator of the quantity C^*_{nk} are of the same sign $(-1)^{k+1}$. *(ii)* can be proved again by the summand-splitting

$$c^*_{nk} = \frac{(q^{n-k+1})_k}{\phi(n, k : q)} q^{\binom{k}{2}} - \frac{(q^{n-k})_{k+1}}{\phi(n, k+1 : q)} q^{\binom{k+1}{2}}$$

and the diagonal-cancelling, noting that $(q^0)_{n+1} = 0$. For proving (iii) it needs only to observe that when $n \to \infty$ (i.e., $q^n \to \infty$) the fraction $(q^{n-k+1})_k/\phi(n, k+1 : q)$ is precisely of the order $O(q^{-n})$.

From the Lemma and by the Toeplitz theorem [3] we obtain the following

Theorem 2. *The linear transformations (arithmetic means) defined by*

$$t_n = \sum_{k=0}^{n} s_k c^*_{nk} \tag{10}$$

are regular provided that the conditions (8) *and* $q > 1$ *are satisfied.*

Remark. It may be of interest to notice that (3) is a limiting case of (10) as $q \to 1^+$. In fact, starting with (10), replacing μ_i by $1 + (1 - q)\lambda_i$ and letting $q \to 1^+$ one may find that $c^*_{nk} \to c_{nk}$, and that the condition (8) just turns to be $\lambda_{k+1} > -k$, which is required for (3). The details may be left to the reader.

References

[1] G. P. Egorychev, *Combinatorial Sums and the Method of Generating Functions,* Krasnoysrsk Gos. Univ., Krasnoyarsk, 1974 (Russian).

[2] G. P. Egorychev, "Integral Representation and the Computation of Combinatorial Sums," *Transl. AMS Monographs,* Vol. 59 (1984), p. 109.

[3] G. H. Hardy, *Divergent Series,* Clarendon Press, Oxford, 1949 (Chap. 3).

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