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GLOBAL ATTRACTIVITY IN A NONAUTONOMOUS DELAY-LOGISTIC EQUATION

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Abstract. Consider the nonautonomous delay-Logistic equation

$$x'(t) = r(t)x(t)[1 - b_1x(t-\tau_1) - b_2x(t-\tau_2)], \quad t \geq 0$$

we obtain sufficient conditions for the positive steady state $x^* = 1/(b_1+b_2)$ to be a global attractor. An application of our result also solves a conjecture of Gopalsamy.

1. Introduction

In this paper, we are concerned with the global attractivity of the nonautonomous delay-Logistic equation

$$x'(t) = r(t)x(t)[1 - b_1x(t-\tau_1) - b_2x(t-\tau_2)], \quad t \geq 0 \tag{1}$$

Throughout this paper, we let the following two hypotheses hold:
(i). $b_1, b_2, \tau_1, \tau_2 \in (0, \infty)$;
(ii). $r(t) \in C([-\tau, \infty), R^+, r(t) > 0$ and $\int_0^\infty r(t)dt = \infty, \tau = \max(\tau_1, \tau_2)$.
Suppose also that the initial conditions for (1) are of the type

$$\begin{aligned} &x(s) = \phi(s) \geq 0, \quad s \in [-\tau, 0], \\ &\phi \in C([-\tau, 0], R^+), \quad \phi(0) > 0. \end{aligned} \tag{2}$$

Eq.(1) with $r(t) \equiv r \in (0, \infty)$, say

$$x'(t) = x(t)[r - a_1x(t-\tau_1) - a_2x(t-\tau_2)], \quad t \geq 0 \tag{3}$$

where $a_1 = rb_1, a_2 = rb_2$.

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The scalar autonomous ordinary differential equation

$$x'(t) = rx(t)\left[1 - \frac{x(t)}{k}\right], \quad k > 0 \tag{4}$$

commonly known as the Logistic equation is most frequently employed in modelling the dynamics of populaton of single species with $x(t)$ denoting the density (or biomass) of the population at time t. An analysis of (4) indicates that its solutions are monotone functions of t and that $\lim_{t\to\infty} x(t) = k$ if $x(0) > 0$. On the other hand, it has been observed, however, that population densities usually have a tendency to fluctuate around an equilibrium, and when there is a convergence to a positive equilibrium, such a convergence is rarely monotonic (See for example Nicholson [5]). To incorporate such oscillations in population model systems. Hutchinson [6] suggested the following modification of (4)

$$x'(t) = rx(t)\left[1 - \frac{x(t-\tau)}{k}\right]. \tag{5}$$

Eq.(5) commonly known as the delay-Logistic equation and has been extensively investigated for the global attractivity (See for example Gopalsamy [3] and the references cited therein).

In an attempt to introduce a environmental negative feedback effect, (5) has been modified and generalized to Eq.(3). For some recent discussions of (3), we refer to Gopalsamy [1,3], Lenhart and Travis [2]. In [1] (also see [3]), by using Liapunov functional method Gopalsamy obtained the following theorem which provides a sufficient condition for the global attractivity of Eq.(3)-(2).

Theorem A. *Assume that* $r, a_1, a_2, \tau_1, \tau_2 \in (0, \infty)$. *Then*

$$r(\tau_1 + \tau_2)\exp[r(\tau_1 + \tau_2)] < 1 \tag{6}$$

implies that all positive solutions of Eq.(3) − (2) *have asymptotic behavior*

$$\underset{t\to\infty}{Lim}\, x(t) = x^*, \quad x^* = r/(a_1 + a_2). \tag{7}$$

In particular, Gopalsamy put forth the following conjecture in [3, p60]:

Conjecture B. Theorem A is also true if condition (6) is relaxed as

$$r(\tau_1 + \tau_2)\exp[r(\tau_1 + \tau_2)] < 3/2. \tag{8}$$

Main aim in this paper is giving sufficient condition for the global attractivity of Eq.(1)-(2). In particular, an application of our result to Eq.(3)-(2) not only solve above Conjecture B but also further improve condition (8).

2. Main Results

The following theorem provides a sufficient condition for the global attractivity of Eq.(1)-(2).

Theorem 1. *Assume that condition (i) and (ii) hold. Then*

$$\limsup_{t\to\infty}\int_{t-\tau}^{t} r(s)ds \le 1, \quad \tau = \max(\tau_1,\tau_2) \tag{9}$$

implies that all positive solutions of Eq.(1) − (2) have asymptotic behavior:

$$\lim_{t\to\infty} x(t) = x^*, \quad x^* = 1/(b_1+b_2).$$

Proof. Let $x(t)$ be any a positive solution of (1)-(2). There are only two cases to consider.

Case 1. $x(t)$ is nonoscillatory about x^*. We shall assume that $x(t) \ge x^*$ eventually. The case where $x(t) \le x^*$ eventually is similar and will be omitted. Then it follows from (1) that $x(t)$ is eventually decreasing and so the limit $\lim_{t\to\infty} x(t) = L$ exists and is finite. Thus, from (1) we see that eventually

$$x'(t) \le r(t)x(t)[1 - L(b_1+b_2)],$$

which, together with $L \ge x^*$ and (ii), implies that $L = x^*$. The proof is complete for the case 1.

Case 2. $x(t)$ is oscillatory about x^*. Let $\{t_n\}$ be a sequence of zeros of the $x(t)-x^*$ such that $t_n \to \infty$ as $n \to \infty$ and that $x(t) \ge x^*, t \in [t_{2n-1}, t_{2n}]$, and $x(t) \le x^*, t \in [t_{2n}, t_{2n+1}], n = 1,2,\ldots$. Also let $t_n^* \in (t_{2n-1}, t_{2n}), s_n^* \in (t_{2n}, t_{2n+1})$ be such that

$$x(t_n^*) = \max(x(t) : t_{2n-1} \le t \le t_{2n}),$$
$$x(s_n^*) = \min(x(t) : t_{2n} \le t \le t_{2n+1}).$$

Then for $n = 1,2,\ldots, x(t_n^*) > x^*$ and $x'(t_n^*) = 0$. While $x(s_n^*) < x^*$ and $x'(s_n^*) = 0$. Thus we obtain by (1)

$$1 - b_1x(t_n^* - \tau_1) - b_2x(t_n^* - \tau_2) = 0,$$

which shows that there exists at least one $i_0 \in \{1,2\}$ such taht $x(t_n^* - \tau_{i_0}) \le x^*$ and so $x(t) - x^*$ has at least one zero on $[t_n^* - \tau, t_n^*]$. Let $\delta_n \in [t_n^* - \tau, t_n^*]$ be such that $x(\delta_n) = x^*$ and $x(t) > x^*$ for $t \in (\delta_n, t_n^*]$. Similarly, there exist $r_n \in [s_n^* - \tau, s_n^*]$ such taht $x(r_n) = x^*$ and $x(t) < x^*$ for $t \in (r_n, s_n^*]$. In the following discussion, for convenience, when we write a functional (or sequential) inequality without specifying its domain of validity. We assume that it holds for all sufficiently large t (or n).

Now we rewrite Eq.(1) in the form

$$\frac{d}{dt}\ln\,[x(t)] = r(t)[1 - b_1 x(t-\tau_1) - b_2 x(t-\tau_2)] \tag{10}$$

and by integrating the both sides of (10) from δ_n to t_n^* we find

$$\ln\,[x(t_n^*)/x^*] < \int_{\delta_n}^{t_n^*} r(s)ds \le \int_{t_n^*-\tau}^{t_n^*} r(s)ds \le 1,$$

i.e., $x(t_n^*) < x^* e := M_0$. Set $m_0 = 0$, then

$$m_0 < x(t) < m_0. \tag{11}$$

Again integrating (10) from r_n to s_n^* and by using (11) we have

$$\ln\,[x(s_n^*)/x^*] > \int_{r_n}^{s_n^*} r(s)[1 - b_1 M_0 - b_2 M_0]ds$$

$$\ge [1 - M_0(b_1 + b_2)] \int_{s_n^*-\tau}^{s_n^*} r(s)ds \ge 1 - M_0(b_1 + b_2),$$

i.e., $x(s_n^*) > x^* \exp[1 - M_0(b_1 + b_2)] := m_1$, and so

$$x(t) > m_1. \tag{12}$$

Similarly, integrating (10) from δ_n to t_n^* and using (12) we get

$$\ln\,[x(t_n^*)/x^*] < [1 - m_1(b_1 + b_2)] \int_{t_n^*-\tau}^{t_n^*} r(s)ds \le 1 - m_1(b_1 + b_2),$$

i.e., $x(t_n^*) < x^* \exp[1 - m_1(b_1 + b_2)] := M_1$, and so

$$x(t) < M_1. \tag{13}$$

Also integrating (10) from r_n to s_n^* and applying (13) we obtain

$$x(s_n^*) > x^* \exp[1 - M_1(b_1 + b_2)] := m_2,$$

which yields,

$$x(t) > m_2. \tag{14}$$

Therefore, by the mathematical induction we can get in general that

$$m_n < x(t) < M_n, \tag{15}$$

where $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ are defined as

$$\begin{aligned} &m_0 = 0, m_n = x^* \exp[1 - M_{n-1}(b_1 + b_2)], \quad n = 1, 2, \dots \\ &M_0 = x^* e, M_n = x^* \exp[1 - m_n(b_1 + b_2)], \quad n = 1, 2, \dots \end{aligned} \tag{16}$$

Clearly, $M_1 < M_0$, which implies $m_1 > m_0$. In general, by the induction, it is easy to prove that

$$M_0 > M_1 > \cdots > M_n > M_{n+1} > \cdots > x^*,$$
$$m_0 < m_1 < \cdots < m_n < m_{n+1} < \cdots < x^*.$$

Set $m = \lim_{n\to\infty} m_n, M = \lim_{n\to\infty} M_n$. Then $m \le x^* \le M$.

Next, by taking limit on (16) we get

$$m = x^* \exp[1 - M(b_1 + b_2)], \quad M = x^* \exp[1 - m(b_1 + b_2)], \tag{17}$$

which shows that the system of equations

$$\begin{cases} u = x^* \exp[1 - (b_1 + b_2)v] \\ v = x^* \exp[1 - (b_1 + b_2)u] \end{cases} \tag{18}$$

has a solution $u = m, v = M$. Clearly, $u = v = x^*$ also is a solution of (18). Now we will prove that (18) has only a unique solution $u = v = x^*$ in the region $D = \{(u, v) : u \ge x^*, v \le x^*\}$. To this end, we rewrite (18) in the form

$$\begin{cases} v = x^* \exp[1 - (b_1 + b_2)u] \\ v = -\dfrac{1}{b_1 + b_2} \ln \dfrac{u}{x^* e} \end{cases} \tag{19}$$

and set

$$f(u) = x^* \exp[1 - (b_1 + b_2)u] + \frac{1}{b_1 + b_2} \ln \frac{u}{x^* e}.$$

Then, it suffices to prove that $f(u) = 0$ has only a unique solution $u = x^*$ on $[x^*, \infty)$. Since

$$\begin{aligned} f'(u) =& \{1 - x^*(b_1 + b_2)^2 u \exp[1 - (b_1 + b_2)u]\}/(b_1 + b_2)u \\ :=& g(u)/(b_1 + b_2)u, \end{aligned}$$

it follows that for $u \ge x^*$, $f'(u)$ and $g(u)$ are of uniform sign. From

$$g'(u) = x^*(b_1 + b_2)^2[(b_1 + b_2)u - 1] \exp[1 - (b_1 + b_2)u]. \tag{20}$$

we see that $x^* = 1/(b_1 + b_2)$ is a unique root of $g'(u) = 0$, and $g'(u) > 0$ for $u > x^*$. Noting $g(x^*) = 0$, which yields $g(u) > 0$ for $u > x^*$. Therefore, we have $f'(x^*) = 0$ and $f'(u) > 0$ for $u > x^*$. This implies that $f(u) = 0$ has only a unique solution $u = x^*$ on

$[x^*, \infty)$ and so $M = m = x^*$. Now by (15) we obtain $\lim_{t\to\infty} x(t) = x^*$. The proof is complete for the case 2.

Combining the cases 1 and 2 we complete the proof of Theorem 1.

Corollary 1. *Theorem A is true if condition* (6) *is replaced by*

$$r\tau \leq 1, \quad \tau = \max(\tau_1, \tau_2). \tag{21}$$

In fact, set $r(t) \equiv r > 0, rb_1 = a_1, rb_2 = a_2$. Then Eq.(1) reduces to Eq.(3) and condition (9) reduces to (21). The conclusion of the corollary 1 follows from Theorem 1.

Remark 1. Corollary 1 not only gives a proof of Conjecture B but also further improve condition (8). In fact, from the fact that $r\tau > 1$ implies the $r(\tau_1+\tau_2)\exp[r(\tau_1+\tau_2)] > r\tau\exp(r\tau) > e > 3/2$, it follows that (8) implies (21). On the other hand, if we choose $r = \tau_1 = \tau_2 = 1$, then (21) holds, but(8) does not satisfy.

Remark 2. The results of this paper are extendble to the delay-Logistic equation of the form

$$x'(t) = r(t)x(t)\left[1 - \sum_{i=1}^{n} b_i x(t-\tau_i)\right].$$

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