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ON THE EPSTEIN ZETA FUNCTION

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Abstract. The Epstein zeta function $Z(s)$ is defined for Re $s > 1$ by

$$Z(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(am^2 + bmn + cn^2)^s},$$

where a, b, c are real numbers with $a > 0$ and $b^2 - 4ac < 0$. $Z(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$. Simple proofs of the functional equation and of the Kronecker "Grenz-formel" for $Z(s)$ are given. The value of $Z(k)(k = 2, 3, \ldots)$ is determined in terms of infinite series of the form

$$\sum_{n=1}^{\infty} \frac{\cot^r n\pi\tau}{n^{2k-1}} (r = 1, 2, \ldots, k),$$

where $\tau = (b + \sqrt{b^2 - 4ac})/2a$.

1. Introduction

Let a, b and c be real numbers with $a > 0$ and $D = 4ac - b^2 > 0$, so that

$$Q(u, v) = au^2 + buv + cv^2 \tag{1.1}$$

is a positive-definite binary quadratic form of discriminant -D.

The Epstein zeta function $Z(s)$ is defined by the double series

$$Z(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{Q(m,n)^s}, \tag{1.2}$$

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where $s = \sigma + it$ and σ, t are real numbers with $\sigma > 1$.

Since $Q(u,v) \geq \lambda(u^2+v^2)$ with

$$\lambda = \frac{1}{2}\left(a + c - \sqrt{(a-c)^2+b^2}\right) > 0,$$

for all real numbers u and v, the series (1.2) converges absolutely for $\sigma > 1$ and uniformly in every half plane $\sigma \geq 1 + \epsilon (\epsilon > 0)$. Thus $Z(s)$ is an analytic function of s for $\sigma > 1$. Furthermore, the function $Z(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$ and satisfies the functional equation

$$\left(\frac{\sqrt{D}}{2\pi}\right)^s \Gamma(s) Z(s) = \left(\frac{\sqrt{D}}{2\pi}\right)^{1-s} \Gamma(1-s) Z(1-s). \tag{1.3}$$

The purpose of this paper is three-fold. First, in §2, by making use of the Poisson summation formula and an integral representation of the Bessel function, we give a proof of the functional equation (1.3), which in the authors' view is simpler than those given in [1], [4], [7]. Secondly, in §3, we deduce, in a very simple manner from the results of §2, the values of A_{-1} and A_0 in the Laurent expansion

$$Z(s) = \frac{A_{-1}}{s-1} + A_0 + A_1(s-1) + \cdots,$$

valid in a neighbourhood of $s = 1$ (the so-called Kronecker "Grenz-formel"). Again we believe our proof to be more direct than those in [3], [4] and [5]. Finally , in §4, we determine the value of $Z(k)$ for any positive integer $k \geq 2$ in terms of series of the type $\sum_{n=1}^{\infty} \frac{\cot^r n\pi\tau}{n^{2k-1}}$, where $\tau = (b + i\sqrt{D})/2a$ and $r = 1, \ldots, k$. The reader should compare our result with that of Smart [6].

2. The Functional Equation of $Z(s)$.

Setting

$$x = \frac{b}{2a}, \quad y = \frac{\sqrt{D}}{2a}, \quad \tau = x + iy = \frac{b + i\sqrt{D}}{2a}, \tag{2.1}$$

we have

$$\tau + \overline{\tau} = 2x = \frac{b}{a}, \quad \tau\overline{\tau} = \frac{b^2 + D}{4a^2} = \frac{c}{a},$$

so that

$$Q(m,n) = am^2 + bmn + cn^2 = a(m + n\tau)(m + n\overline{\tau}) = a|m + n\tau|^2 \tag{2.2}$$

and

$$Z(s) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{a^s |m + n\tau|^{2s}}, \quad \sigma > 1.$$

Separating the term with $n = 0$, we obtain

$$Z(s) = \frac{2}{a^s} \sum_{m=1}^{\infty} \frac{1}{m^{2s}} + \frac{2}{a^s} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{|m+n\tau|^{2s}}, \quad \sigma > 1. \tag{2.3}$$

In order to evaluate the second term in (2.3), we apply the Poisson summation formula [8, p.17]

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) \cos 2m\pi u \, du \tag{2.4}$$

to the function $f(t) = \frac{1}{|t+\tau|^{2s}}$ and obtain

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} \frac{1}{|m+\tau|^{2s}} &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos 2m\pi u}{|u+\tau|^{2s}} du \\
&= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos 2m\pi u}{\{(u+x)^2+y^2\}^s} du = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\cos 2m\pi(t-x)}{(t^2+y^2)^s} dt \\
&= \sum_{m=-\infty}^{\infty} \cos 2m\pi x \int_{-\infty}^{\infty} \frac{\cos 2m\pi t}{(t^2+y^2)^s} dt,
\end{aligned}$$

since the integrals involving the sine function vanish, that is

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty} \frac{1}{|m+\tau|^{2s}} \\
=&\frac{2}{y^{2s-1}} \int_0^{\infty} \frac{1}{(1+t^2)^s} dt + \frac{2^2}{y^{2s-1}} \sum_{m=1}^{\infty} \cos 2m\pi x \int_0^{\infty} \frac{\cos 2m\pi yt}{(1+t^2)^s} dt, \quad \sigma > 1.
\end{aligned} \tag{2.5}$$

Next, we evaluate the two integrals appearing in (2.5). Making the substitution $u = \frac{t^2}{1+t^2}$, we have

$$\frac{1}{1+t^2} = 1-u, \quad du = \frac{2tdt}{(1+t^2)^2} = 2u^{1/2}(1-u)^{3/2} dt,$$

and

$$\begin{aligned}
\int_0^{\infty} \frac{dt}{(1+t^2)^s} &= \frac{1}{2} \int_0^1 (1-u)^{s-3/2} u^{-1/2} du = \frac{1}{2} B\left(s-\frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{\Gamma(s-\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(s)} = \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{2\Gamma(s)}.
\end{aligned} \tag{2.6}$$

From an integral representation of the Bessel function [2, p.140]

$$K_\nu(y) = \frac{1}{\sqrt{\pi}} \left(\frac{2}{y}\right)^{\nu} \Gamma(\nu+\frac{1}{2}) \int_0^{\infty} \frac{\cos yt}{(1+t^2)^{\nu+\frac{1}{2}}} dt, \quad y > 0, \quad \mathrm{Re}\nu > -\frac{1}{2}, \tag{2.7}$$

we have

$$\int_0^\infty \frac{\cos 2m\pi yt\,dt}{(1+t^2)^s} = \frac{\sqrt{\pi}(m\pi y)^{s-\frac{1}{2}}}{\Gamma(s)} K_{s-\frac{1}{2}}(2m\pi y), \quad \sigma > \frac{1}{2}. \tag{2.8}$$

Putting (2.6) and (2.8) into (2.5), we obtain

$$\sum_{m=-\infty}^{\infty} \frac{1}{|m+\tau|^{2s}}$$
$$= \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{y^{2s-1}\Gamma(s)} + \frac{4\sqrt{\pi}}{y^{2s-1}\Gamma(s)} \sum_{m=1}^{\infty} (m\pi y)^{s-\frac{1}{2}} \cos(2m\pi x) K_{s-\frac{1}{2}}(2m\pi y), \quad \sigma > 1,$$

so that for $n \geq 1$

$$\sum_{m=-\infty}^{\infty} \frac{1}{|m+n\tau|^{2s}}$$
$$= \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{n^{2s-1}y^{2s-1}\Gamma(s)} + \frac{4\sqrt{\pi}}{n^{2s-1}y^{2s-1}\Gamma(s)} \sum_{m=1}^{\infty} (mn\pi y)^{s-\frac{1}{2}} \cos(2mn\pi x) K_{s-\frac{1}{2}}(2mn\pi y). \tag{2.9}$$

Then, from (2.9) and (2.3), we have

$$Z(s) = 2a^{-s}\zeta(2s) + 2a^{-s}y^{1-2s}\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}\zeta(2s-1)$$
$$+ \frac{8a^{-s}y^{\frac{1}{2}-s}\pi^s}{\Gamma(s)} \sum_{n=1}^{\infty} n^{1-2s} \sum_{m=1}^{\infty} (mn)^{s-\frac{1}{2}} \cos(2mn\pi x) K_{s-\frac{1}{2}}(2mn\pi y). \tag{2.10}$$

Collecting the terms with $mn = k$, we obtain

$$Z(s) = 2a^{-s}\zeta(2s) + 2a^{-s}y^{1-2s}\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}\zeta(2s-1)$$
$$+ \frac{8a^{-s}y^{\frac{1}{2}-s}\pi^s}{\Gamma(s)} \sum_{k=1}^{\infty} \left(\sum_{n|k} n^{1-2s}\right) k^{s-\frac{1}{2}} \cos(2k\pi x) K_{s-\frac{1}{2}}(2k\pi y), \tag{2.10}$$

that is

$$Z(s) = 2a^{-s}\zeta(2s) + 2a^{-s}y^{1-2s}\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}\zeta(2s-1) + \frac{2a^{-s}y^{\frac{1}{2}-s}\pi^s}{\Gamma(s)} H(s), \tag{2.11}$$

where

$$H(s) = 4\sum_{k=1}^{\infty} \sigma_{1-2s}(k) k^{s-\frac{1}{2}} \cos(2k\pi x) K_{s-\frac{1}{2}}(2k\pi y), \tag{2.12}$$

and $\sigma_\nu(k)$ denotes the sum of the ν-th powers of the divisors of k, that is,

$$\sigma_\nu(k) = \sum_{d|k} d^\nu = \sum_{d|k} \left(\frac{k}{d}\right)^\nu .$$

The formula (2.11) provides the analytic continuation of $Z(s)$. In fact, the first two terms on the right-side of (2.11) have removable singularities at $s = \frac{1}{2} - n$ $(n = 0, 1, 2, \ldots)$ because at $s = \frac{1}{2}$ the pole of $\zeta(2s)$ is cancelled by the pole of $\Gamma(s - \frac{1}{2})$ and at $s = \frac{1}{2} - n$ $(n = 1, 2, \ldots)$ the pole of $\Gamma(s - \frac{1}{2})$ is cancelled by the zero of $\zeta(2s-1)$ $(\zeta(-2n) = 0,$ $n = 1, 2, \ldots)$; besides, it is not difficult to prove that $H(s)$ is an entire function [1]. Thus, it follows the $Z(s)$ has a continuation in the whole finite complex plane except for a simple pole at $s = 1$.

Now, we write (2.11) in another form:

$$\left(\frac{ay}{\pi}\right)^s \Gamma(s)Z(s) = 2\left(\frac{y}{\pi}\right)^s \Gamma(s)\zeta(2s) + 2y^{1-s}\pi^{\frac{1}{2}-s}\Gamma(s-\frac{1}{2})\zeta(2s-1) + 2y^{\frac{1}{2}}H(s). \quad (2.13)$$

By the functional equation for the Riemann zeta function

$$\zeta(2s-1) = 2(2\pi)^{2s-2}\sin(s-\frac{1}{2})\pi\Gamma(2-2s)\zeta(2-2s)$$

and the basic properties of the gamma function

$$\begin{aligned}\Gamma(2-2s) &= \frac{\Gamma(1-s)\Gamma(\frac{3}{2}-s)}{\sqrt{\pi}2^{2s-1}} = \frac{\Gamma(1-s)}{\sqrt{\pi}2^{2s-1}} \cdot \frac{\pi}{\Gamma(s-\frac{1}{2})\sin(s-\frac{1}{2})\pi} \\ &= \frac{\sqrt{\pi}\Gamma(1-s)}{2^{2s-1}\Gamma(s-\frac{1}{2})\sin(s-\frac{1}{2})\pi}\end{aligned}$$

we have

$$\pi^{\frac{1}{2}-s}\Gamma(s-\frac{1}{2})\zeta(2s-1) = \pi^{s-1}\Gamma(1-s)\zeta(2-2s)$$

and

$$\left(\frac{ay}{\pi}\right)^s \Gamma(s)Z(s) = 2\left(\frac{y}{\pi}\right)^s \Gamma(s)\zeta(2s) + 2\left(\frac{y}{\pi}\right)^{1-s} \Gamma(1-s)\zeta(2-2s) + 2y^{\frac{1}{2}}H(s). \quad (2.14)$$

Recalling the following facts:

$$K_{-\nu}(y) = K_\nu(y)$$

(see [2, p.110]) and

$$k^{-\frac{\nu}{2}}\sigma_\nu(k) = k^{-\frac{\nu}{2}} \sum_{d|k} d^\nu = k^{-\frac{\nu}{2}} \sum_{d|k} \left(\frac{k}{d}\right)^\nu = k^{\frac{\nu}{2}}\sigma_{-\nu}(k),$$

we have from (2.12)

$$H(s) = H(1-s). \tag{2.15}$$

If we put

$$\phi(s) = \left(\frac{ay}{\pi}\right)^s \Gamma(s) Z(s) \tag{2.16}$$

then from (2.14) and (2.15) we obtain

$$\phi(s) = \phi(1-s). \tag{2.17}$$

Since $ay = \sqrt{D}/2$ (from (2.1)), (2.17) can be rewritten in the following form

$$\left(\frac{\sqrt{D}}{2\pi}\right)^s \Gamma(s) Z(s) = \left(\frac{\sqrt{D}}{2\pi}\right)^{1-s} \Gamma(1-s) Z(1-s), \tag{2.18}$$

which is the functional equation of $Z(s)$.

3. Determination of A_{-1} and A_0.

As $Z(s)$ is an analytic function in the finite complex plane except for a simple pole at $s=1$, $Z(s)$ has a Laurent expansion

$$Z(s) = \frac{A_{-1}}{s-1} + A_0 + A_1(s-1) + \cdots \tag{3.1}$$

valid in a neighbourhood of $s=1$. In this neighbourhood of $s=1$, we have the following expansions:

$$\zeta(2s-1) = \frac{1}{2(s-1)} + \gamma + \cdots,$$

where γ denotes Euler's constant,

$$\begin{aligned} \frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)} &= \pi + \pi^{\frac{1}{2}} \left[\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\right]' \Big|_{s=1} (s-1) + \cdots \\ &= \pi + \pi^{\frac{1}{2}} \left(\psi(\frac{1}{2}) + \gamma\right) \Gamma(\frac{1}{2})(s-1) + \cdots \qquad \left(\text{where } \psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}\right) \\ &= \pi - 2\pi \log 2 \cdot (s-1) + \cdots, \end{aligned}$$

since $\psi(1) = -\gamma$ and $\psi(\frac{1}{2}) = -\gamma - 2\log 2$;

$$2a^{-s}y^{1-s} = \frac{2}{ay}[1 - \log(ay^2)(s-1) + \cdots];$$

and

$$\begin{aligned}
&2a^{-s}y^{1-2s}\frac{\Gamma(s-\frac{1}{2})\sqrt{\pi}}{\Gamma(s)}\zeta(2s-1)\\
=&\frac{2\pi}{ay}[1-\log(ay^2)(s-1)+\cdots][1-2\log 2(s-1)+\cdots]\left[\frac{1}{2(s-1)}+\gamma+\cdots\right]\\
=&\frac{4\pi}{\sqrt{D}}[1-\log(ay^2)(s-1)+\cdots]\left[\frac{1}{2(s-1)}+(\gamma-\log 2)+\cdots\right]\\
=&\frac{4\pi}{\sqrt{D}}\left[\frac{1}{2(s-1)}+\left(\gamma-\log 2-\frac{1}{2}\log(ay^2)\right)+\cdots\right]\\
=&\frac{2\pi}{\sqrt{D}}\cdot\frac{1}{s-1}+\frac{2\pi}{\sqrt{D}}\left(2\gamma-\log\frac{D}{a}\right)+\cdots. \qquad (3.2)
\end{aligned}$$

From (3.2) and (2.11), we obtain

$$Z(s)=\frac{\pi^2}{3a}+\frac{2\pi}{\sqrt{D}}\cdot\frac{1}{s-1}+\frac{2\pi}{\sqrt{D}}\left(2\gamma-\log\frac{D}{a}\right)+2a^{-1}y^{-\frac{1}{2}}\pi H(1)+O(|s-1|). \qquad (3.3)$$

Next, we evaluate the fourth term on the right side of (3.3) which contains $H(1)$. We have

$$\begin{aligned}
2a^{-1}y^{-\frac{1}{2}}\pi H(1)=&8a^{-1}y^{-\frac{1}{2}}\pi\sum_{k=1}^{\infty}\sigma_{-1}(k)k^{\frac{1}{2}}\cos(2\pi kx)K_{\frac{1}{2}}(2\pi ky)\\
=&8a^{-1}y^{-1}\sum_{k=1}^{\infty}\sigma_{-1}(k)\cos(2\pi kx)\int_0^{\infty}\frac{\cos 2\pi kyt}{1+t^2}dt \quad (\text{by } (2.7))\\
=&\frac{8\pi}{\sqrt{D}}\sum_{k=1}^{\infty}\sigma_{-1}(k)\cos(2\pi kx)e^{-2\pi ky} \quad \left(\text{as } \int_0^{\infty}\frac{\cos lt}{1+t^2}dt=\frac{\pi}{2}e^{-l}\right).
\end{aligned}$$

If we set $q=e^{\pi i\tau}$ (so that $|q|=e^{-\pi y}<1$ as $y>0$) then $q^{2k}+\bar{q}^{2k}=2e^{-2k\pi y}\cos 2k\pi x$ and

$$\begin{aligned}
2a^{-1}y^{-\frac{1}{2}}\pi H(1)=&\frac{4\pi}{\sqrt{D}}\left(\sum_{k=1}^{\infty}\sigma_{-1}(k)q^{2k}+\sum_{k=1}^{\infty}\sigma_{-1}(k)\bar{q}^{2k}\right)\\
=&-\frac{4\pi}{\sqrt{D}}\log\prod_{k=1}^{\infty}(1-q^{2k})(1-\bar{q}^{2k})\\
=&-\frac{4\pi}{\sqrt{D}}\log\prod_{k=1}^{\infty}|1-q^{2k}|^2,
\end{aligned}$$

and from (3.3) we have

$$Z(s)=\frac{\pi^2}{3a}+\frac{2\pi}{\sqrt{D}}\cdot\frac{1}{s-1}+\frac{2\pi}{\sqrt{D}}\left(2\gamma-\log\frac{D}{a}\right)-\frac{4\pi}{\sqrt{D}}\log\prod_{k=1}^{\infty}|1-q^{2k}|^2+O(|s-1|). \quad (3.4)$$

The Dedekind eta function $\eta(z)$ is given by

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \quad z = x + iy, \quad y > 0. \tag{3.5}$$

Hence we have

$$\log \prod_{n=1}^{\infty} |1 - q^{2n}|^2 = \log |\eta(\tau)|^2 + \frac{\pi y}{6} = 2 \log |\eta(\tau)| + \frac{\pi \sqrt{D}}{12a}$$

giving another form of (3.4):

$$Z(s) = \frac{2\pi}{\sqrt{D}} \cdot \frac{1}{s-1} + \frac{2\pi}{\sqrt{D}} \left(2\gamma - \log \frac{D}{a}\right) - \frac{8\pi}{\sqrt{D}} \log |\eta(\tau)| + O(|s-1|). \tag{3.6}$$

We have shown that

$$A_{-1} = \frac{2\pi}{\sqrt{D}}, \quad A_0 = \frac{4\pi\gamma}{\sqrt{D}} - \frac{2\pi}{\sqrt{D}} \log \frac{D}{a} - \frac{8\pi}{\sqrt{D}} \log \left| \eta\left(\frac{b + i\sqrt{D}}{2a}\right) \right|. \tag{3.7}$$

4. Evaluation of $Z(k)$ for any positive integer $k \geq 2$.

We need the following lemma.

Lemma. *For any non-negative integer k and any non-negative real number λ,*

$$I_k(\lambda) = \int_0^{\infty} \frac{\cos 2\pi\lambda x}{(1+x^2)^{k+1}} dx = \frac{\pi}{2^{2k+1}} e^{-2\pi\lambda} \sum_{j=0}^{k} \binom{2k-j}{k} (4\pi\lambda)^j / j!. \tag{4.1}$$

Proof. Let C_R denote the contour in the upper half of the complex z-plane consisting of the semi-circle $|z| = R$ and the real axis from $z = -R$ to $z = R$. Applying Cauchy's residue theorem to the integral of the function $\frac{e^{2\pi i\lambda z}}{(1+z^2)^{k+1}}$ along the contour C_R, and then letting $R \to +\infty$, we obtain

$$\begin{aligned}
I_k(\lambda) &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{2\pi i\lambda x}}{(1+x^2)^{k+1}} dx = \frac{1}{2} \cdot 2\pi i \cdot \frac{1}{k!} \frac{d^{k+1}}{dz^{k+1}} \left\{ \frac{e^{2\pi i\lambda z}}{(z+i)^{k+1}} \right\}_{z=i} \\
&= \frac{\pi i}{k!} \sum_{j=0}^{k} \binom{k}{j} (e^{2\pi i\lambda z})^{(j)}_{z=i} \left(\frac{1}{(z+i)^{k+1}} \right)^{(k-j)}_{z=i} \\
&= \frac{\pi i}{k!} \sum_{j=0}^{k} \binom{k}{j} (2\pi i\lambda)^j e^{-2\pi\lambda} \frac{(k+1)(k+2)\cdots(k+k-j)}{(2i)^{2k-j+1}} (-1)^{k-j} \\
&= \frac{\pi}{2^{2k+1}} e^{-2\pi\lambda} \sum_{j=0}^{k} \binom{2k-j}{k} (4\pi\lambda)^j / j!.
\end{aligned}$$

Now we start the evaluation of $Z(k+1), k \geq 1$. First, from (2.5), we have

$$\sum_{m=-\infty}^{\infty} \frac{1}{|m+\tau|^{2k+2}} = \frac{2}{y^{2k+1}} \int_0^{\infty} \frac{dt}{(1+t^2)^{k+1}} + \frac{4}{y^{2k+1}} \sum_{m=1}^{\infty} \cos(2m\pi x) I_k(my).$$

Then, by (2.6) and the lemma, we deduce that

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty} \frac{1}{|m+\tau|^{2k+2}} \\
=&\frac{\Gamma(k+\frac{1}{2})\sqrt{\pi}}{y^{2k+1}k!} + \frac{4\pi}{(2y)^{2k+1}} \sum_{j=0}^{k} \binom{2k-j}{k} \frac{1}{j!} \sum_{m=1}^{\infty} (4\pi my)^j \cos(2m\pi x) e^{-2m\pi y} \\
=&\frac{2\pi}{(2y)^{2k+1}} \binom{2k}{k} + \frac{4\pi}{(2y)^{2k+1}} \sum_{j=0}^{k} \binom{2k-j}{k} \frac{1}{j!} \sum_{m=1}^{\infty} (4\pi my)^j \cos(2m\pi x) e^{-2m\pi y}.
\end{aligned}$$

Putting the above equality into (2.3), we obtain

$$\begin{aligned}
&\frac{1}{2} a^{k+1} Z(k+1) \\
=&\zeta(2k+2) + \frac{2\pi}{(2y)^{2k+1}} \binom{2k}{k} \zeta(2k+1) \\
&+ \frac{4\pi}{(2y)^{2k+1}} \sum_{j=0}^{k} \binom{2k-j}{k} \frac{1}{j!} (4\pi y)^j \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} \sum_{m=1}^{\infty} (mn)^j \cos(2mn\pi x) e^{-2mn\pi y}.
\end{aligned}$$

Since

$$\begin{aligned}
\cos(2mn\pi x) e^{-2mn\pi y} =&\frac{1}{2}(e^{2mn\pi i\tau} + e^{-2mn\pi i\bar{\tau}}) \\
=&\frac{1}{2}(e(mn\tau) + e(-mn\bar{\tau})),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{1}{2} a^{k+1} Z(k+1) \\
=&\zeta(2k+2) + \frac{2\pi}{(2y)^{2k+1}} \binom{2k}{k} \zeta(2k+1) \\
&+ \frac{2\pi}{(2y)^{2k+1}} \sum_{j=0}^{k} \binom{2k-j}{k} \frac{1}{j!} (4\pi y)^j \sum_{n=1}^{\infty} \frac{1}{n^{2k+1}} \sum_{m=1}^{\infty} (mn)^j (e(mn\tau) + e(-mn\bar{\tau})). \quad (4.2)
\end{aligned}$$

In view of the fact that if $\operatorname{Im} \tau > 0$,

$$\begin{cases} \sum_{m=1}^{\infty} e(m\tau) = \frac{i}{2} \cot \pi\tau - \frac{1}{2}, \\ \sum_{m=1}^{\infty} e(-m\bar{\tau}) = \frac{-i}{2} \cot \pi\bar{\tau} - \frac{1}{2}, \end{cases} \quad (4.3)$$

we have

$$\sum_{m=1}^{\infty}(e(m\tau)+(-m\bar{\tau}))=\frac{i}{2}(\cot\pi\tau-\cot\pi\bar{\tau})-1. \tag{4.4}$$

From (4.2) and (4.4), we have

$$\begin{aligned}
&\frac{1}{2}a^{k+1}Z(k+1)\\
=&\zeta(2k+2)+\frac{2\pi}{(2y)^{2k+1}}\binom{2k}{k}\left[\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\sum_{m=1}^{\infty}((e(mn\tau)+e(-mn\bar{\tau}))\right]\\
&+\frac{2\pi}{(2y)^{2k+1}}\sum_{j=1}^{k}\binom{2k-j}{k}\frac{1}{j!}(4\pi y)^j\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\sum_{m=1}^{\infty}(mn)^j(e(mn\tau)+e(-mn\bar{\tau}))\\
=&\zeta(2k+2)+\frac{\pi i}{(2y)^{2k+1}}\binom{2k}{k}\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}(\cot\pi n\tau-\cot\pi n\bar{\tau})\\
&+\frac{2\pi}{(2y)^{2k+1}}\sum_{j=1}^{k}\binom{2k-j}{k}\frac{1}{j!}(4\pi y)^j\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}}\sum_{m=1}^{\infty}(mn)^j(e(mn\tau)+e(-mn\bar{\tau})). \quad (4.5)
\end{aligned}$$

Differentiating both sides of (4.3) j times gives

$$\begin{aligned}
(2\pi i)^j\sum_{m=1}^{\infty}m^je(m\tau)=&\frac{i}{2}(\cot\pi\tau)^{(j)}\\
=&\frac{i}{2}(2\pi)^j(-1)^{[\frac{j}{2}]}\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}(\cot\pi\tau)^{j+1-2l},
\end{aligned}$$

that is

$$\sum_{m=1}^{\infty}m^je(m\tau)=\frac{(-1)^j}{2}i^{j+1}(-1)^{[\frac{j}{2}]}\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}(\cot\pi\tau)^{j+1-2l}, \tag{4.6}$$

where each coefficient C_{j+1-2l} can be expressed in terms of Stirling numbers of the second kind [9, p.37]. From (4.6), we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}m^j(e(m\tau)+e(m\bar{\tau}))\\
=&\frac{1}{2}i^{j+1}(-1)^{[\frac{j}{2}]}\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}\{(-1)^j(\cot\pi\tau)^{j+1-2l}-(\cot\pi\bar{\tau})^{j+1-2l}\}.
\end{aligned} \tag{4.7}$$

From (4.5) and (4.7), we have

$$\frac{1}{2}a^{k+1}Z(k+1) = \zeta(2k+2) + \frac{\pi i}{(2y)^{2k+1}}\binom{2k}{k}\sum_{n=1}^{\infty}\frac{\cot n\pi\tau - \cot n\pi\bar{\tau}}{n^{2k+1}}$$
$$+ \frac{\pi i}{(2y)^{2k+1}}\sum_{j=1}^{k}\binom{2k-j}{k}(-1)^{[\frac{j}{2}]}\frac{1}{j!}i^j(4\pi y)^j\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}$$
$$\cdot\sum_{n=1}^{\infty}\frac{(-1)^j(\cot\pi n\tau)^{j+1-2l} - (\cot\pi n\bar{\tau})^{j+1-2l}}{n^{2k+1}},$$

that is

$$\frac{1}{2}a^{k+1}Z(k+1) = \zeta(2k+2) + \frac{\pi i}{(2y)^{2k+1}}\sum_{j=0}^{k}\binom{2k-j}{k}(-1)^{[\frac{j}{2}]}(4\pi y)^j/j!$$
$$\cdot\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}\sum_{n=1}^{\infty}\frac{(-1)^j(\cot\pi n\tau)^{j+1-2l} - (\cot\pi n\bar{\tau})^{j+1-2l}}{n^{2k+1}}. \tag{4.8}$$

In particular, if $b=0$, then $x=0, \tau = yi = \sqrt{\frac{c}{a}}i, \cot\tau = \cot(yi) = -i\coth y$, and

$$\frac{1}{2}a^{k+1}Z(k+1)$$
$$= \zeta(2k+2) + \frac{2\pi}{(2y)^{2k+1}}\sum_{j=0}^{k}(-1)^{[\frac{j+1}{2}]}\binom{2k-j}{k}(4\pi y)^j/j! \tag{4.9}$$
$$\cdot\sum_{l=0}^{[\frac{j+1}{2}]}C_{j+1-2l}\sum_{n=1}^{\infty}\frac{(\coth n\pi y)^{j+1-2l}}{n^{2k+1}}.$$

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