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NONASSOCIATIVE RINGS WITH A SPECIAL DERIVATION

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Abstract. Let R be a nonassociative ring, N, L and G the left nucleus, right nucleus and nucleus respectively. It is shown that if R is a prime ring with a derivation d such that $ax + d(x) \in G$ where $a \in \mathbf{Z}$, the ring of rational integers, or $a \in G$ with $(ad)(x) = ad(x) = d(ax)$ and $ax = xa$ for all x in R then either R is associative or $ad + d^2 = 2d(R)^2 = 0$. This result is also valid under the weaker hypothesis $ax + d(x) \in N \cap L$ for all x in R for the simple ring case, and we obtain that either R is associative or $((ad + d^2)(R))^2 = 0$ for the prime ring case.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notation for associators : $(x, y, z) = (xy)z - x(yz)$. We shall denote the left nucleus, middle nucleus, right nucleus and nucleus by N, M, L and G respectively. Thus N, M, L and G consists of all elements n such that $(n, R, R) = 0$, $(R, n, R) = 0$, $(R, R, n) = 0$ and $(n, R, R) = (R, n, R) = (R, R, n) = 0$ respectively. An additive mapping d on R is called a derivation if $d(xy) = d(x)y + xd(y)$ for all x, y in R. R is called semiprime if the only ideal of R which squares to zero is the zero ideal. R is called prime if the product of any two nonzero ideals of R is nonzero. R is called simple if R is the only nonzero ideal of R. Clearly, a prime ring is a semiprime ring. If R is a simple ring, then $R^2 = 0$ or $R^2 = R$; in the former case R is commutative and associative. So, if R is a simple ring then we assume that $R^2 = R$. Thus a simple ring is a prime ring. Recently, Suh [3] proved that if R is a prime ring with a derivation d such that $d(R) \subseteq G$ then either R is associative or $d^3 = 0$. We generalized and improved Suh's results as follows:

Theorem A [6]. *If R is a prime ring with a derivation d such that $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$ or $d(R) \subseteq N \cap L$, then either R is associative or $d^2 = 2d = 0$.*

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Theorem B [8]. *If R is a prime ring with a derivation d and there exists a fixed positive integer n such that $d^n(R) \subseteq G$, then either R is associative or* $d^{3n-1} = 0$.

In [7], using the result of [2], we have partially extended Theorem A. The purpose of this note is to prove that if R is a prime ring with a derivation d such that $ax+d(x) \in G$ for all x in R where a is as in the Abstract then either R is associative or $ad+d^2 = 2d(R)^2 = 0$. This result is also valid under the weaker hypothesis $ax + d(x) \in N \cap L$ for all x in R for the simple ring case, and we obtain that either R is associative or $((ad+d^2)(R))^2 = 0$ for the prime ring case. Rings with associators in the nuclei were first studied by Kleinfeld and later by the author. Kleinfeld [1] showed that if R is a semiprime ring such that $(R, R, R) \subseteq G$ and the Abelian group $(R, +)$ has no elements of order 2 then R is associative. In [4], we proved that if R is a simple ring of characteristic not two such that $(R, R, R) \subseteq N \cap M$ or $(R, R, R) \subseteq M \cap L$ then R is associative. In [5], we have proved that if R is a semiprime ring such that $(R, R, R) \subseteq N \cap M$ or $(R, R, R) \subseteq M \cap L$ or $(R, R, R) \subseteq N \cap L$ then $N = M = L$ and $2(R, R, R) = 0$. Thus E. Kleinfeld's result can be improved.

Let R be a ring with a derivation d. A nonempty subset S of R is called d-invariant if $d(S) \subseteq S$.

2. Results

Let R be a nonassociative ring. In every ring one may verify the Teichmüller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z \qquad \text{for all } w, x, y, z \text{ in } R. \quad (1)$$

Suppose that $n \in N$. Then with $w = n$ in (1) we obtain

$$(nx, y, z) = n(x, y, z) \text{ for all } n \text{ in } N, \qquad \text{and all } x, y, z \text{ in } R. \quad (2)$$

Assume that $m \in L$. Then with $z = m$ in (1) we get

$$(w, x, ym) = (w, x, y)m \text{ for all } m \text{ in } L, \qquad \text{and all } w, x, y \text{ in } R. \quad (3)$$

As consequences of (1), (2) and (3), we have that $N, M, L, N \cap M, M \cap L, N \cap L$ and G are associative subrings of R.

Definition 1. Let d be a derivation of R, and $a \in \mathbf{Z}$, the ring of rational integers, or $a \in G$ with $(ad)(x) = ad(x) = d(ax)$ and $ax = xa$ for all x in R.

We assume that R has a derivation d which satisfies

$(*)$ $\qquad ax + d(x) \in A$ for all x in R, where A is a subring of R.

Assume that $x, y \in R$. Using the definition of d, Definition 1 and $(*)$, and noting that A is a subring of R, we get

$$a^2xy + ad(xy) + d(x)d(y) = (ax + d(x))(ay + d(y)) \in A.$$

By Definition 1 and $(*)$, $a^2xy + ad(xy) = a(a(xy)) + d(a(xy)) \in A$. Noting that A is a subring of R, these two relations imply

$$d(x)d(y) \in A \text{ for all } x, y \text{ in } R. \tag{4}$$

Applying Definition 1 and $(*)$, we have

$$(ad + d^2)(x) \cdot y + d(x)d(y) = (ad(x) + d^2(x)) \cdot y + d(x)d(y) = a(d(x)y) + d(d(x)y) \in A.$$

Combining this with (4) yields $(ad + d^2)(x) \cdot y \in A$. Thus we obtain

$$(ad + d^2)(R) \cdot R \subseteq A. \tag{5}$$

By Definition 1, $(*)$ and symmetry, we get

$$R \cdot (ad + d^2)(R) \subseteq A. \tag{6}$$

Using Definition 1 and $(*)$, we have $(ad + d^2)(x) = ad(x) + d(d(x)) \in A$. Hence we obtain

$$(ad + d^2)(R) \subseteq A. \tag{7}$$

Definition 2. The associator ideal I of R is the smallest ideal which contains all associators in R.

Note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (1). Hence we have

$$I = \sum (R, R, R) + (R, R, R)R = \sum (R, R, R) + R(R, R, R). \tag{8}$$

Lemma 1. *Let R be a ring and B an additive subgroup of $(R, +)$. If $B \subseteq G$ and $BR + RB \subseteq N \cap M$ or $M \cap L$ or $N \cap L$, then the ideal E of R generated by B is*

$$E = \sum B + BR + RB + R \cdot BR.$$

Proof. Obviously, E is an additive subgroup of $(R, +)$. By symmetry, we only prove the lemma in case $B \subseteq G$ and $BR + RB \subseteq N \cap M$. Thus we obtain

$$(R \cdot BR)R = R \cdot (BR)R = R \cdot BR^2 \subseteq R \cdot BR$$

and

$$R(R \cdot BR) = R(RB \cdot R) = (R \cdot RB)R = R^2B \cdot R \subseteq RB \cdot R = R \cdot BR.$$

Hence E is an ideal of R.

Lemma 2. *If R is a ring with a derivation d such that $ad + d^2 = 0$, then $2d(R)^2 = 0$.*

Proof. Assume that $x, y \in R$. Because of $ad + d^2 = 0$, we obtain

$$\begin{aligned} -ad(xy) &= d^2(xy) = d^2(x)y + 2d(x)d(y) + xd^2(y) \\ &= -ad(x)y + 2d(x)d(y) - axd(y) = -ad(xy) + 2d(x)d(y). \end{aligned}$$

Thus $2d(x)d(y) = 0$, as desired.

Let $A \subseteq N$. Applying (2), (5) and (7) , we get $(ad + d^2)(R) \cdot (R, R, R) = 0$. By this and (7), we have $(ad + d^2)(R) \cdot ((R, R, R)R) = 0$. Using these and (8), we obtain

$$(ad + d^2)(R) \cdot I = 0 \qquad \text{if} \quad A \subseteq N. \tag{9}$$

Let $A \subseteq L$. Applying (3), (6), (7) and (8), we have

$$I \cdot (ad + d^2)(R) = 0 \qquad \text{if} \quad A \subseteq L. \tag{10}$$

Theorem 1. *If R is a simple ring with a derivation d such that $ax + d(x) \in N \cap L$ for all x in R, then either R is associative or $ad + d^2 = 2d(R)^2 = 0$.*

Proof. If $I = 0$, then R is associative. Assume that $I \neq 0$. By the simplicity of R, we get $I = R$. By (9) and (10), we have $(ad+d^2)(R) \cdot R = 0$ and $R \cdot (ad+d^2)(R) = 0$. Thus, the ideal of R generated by $(ad+d^2)(R)$ is $\sum(ad+d^2)(R)$. Hence, $\sum(ad+d^2)(R) \cdot R = 0$ implies $\sum(ad + d^2)(R) = 0$. By Lemma 2, $2d(R)^2 = 0$, as desired.

Theorem 2. *If R is a prime ring with a derivation d such that $ax + d(x) \in N \cap L$ for all x in R, then either R is associative or $((ad + d^2)(R))^2 = 0$.*

Proof. Let $f = ad+d^2$. By the hypothesis, (5), (6) and (7), we have $f(R) \subseteq N \cap L$, $f(R)R \subseteq N \cap L$ and $Rf(R) \subseteq N \cap L$. Applying these, and with $x \in f(R)$ and $y \in f(R)$, and with $x \in f(R)$ and $y \in f(R)R$, and with $x \in Rf(R)$ and $y \in f(R)$ in (1) respectively, we all get $xy \in M$. Combining the above results, we obtain $f(R)^2 \subseteq G$, $f(R) \cdot f(R)R \subseteq G$ and $Rf(R) \cdot f(R) \subseteq G$. Using these and Lemma 1, we see that the ideal C of R generated by $f(R)^2$ is

$$C = \sum f(R)^2 + f(R)^2R + Rf(R)^2 + R \cdot f(R)^2R.$$

By (9), we have $C \cdot I = 0$. Thus by the primeness of R, this implies $C = 0$ or $I = 0$. Hence either R is associative or $((ad + d^2)(R))^2 = 0$, as desired.

Theorem 3. *If R is a prime ring with a derivation d such that $ax+d(x) \in G$ for all x in R, then either R is associative or $ad + d^2 = 2d(R)^2 = 0$.*

Proof. Let $f = ad + d^2$. By the hypothesis, (5), (6) and (7), we get $f(R) \subseteq G$, $f(R)R \subseteq G$ and $Rf(R) \subseteq G$. Using these and Lemma 1, we have that the ideal F of R generated by $f(R)$ is

$$F = \sum f(R) + f(R)R + Rf(R) + R \cdot f(R)R.$$

By (9), we obtain $F \cdot I = 0$. Thus by the primeness of R, this implies $F = 0$ or $I = 0$. Hence either R is associative or $ad + d^2 = 0$. By Lemma 2, $2d(R)^2 = 0$. This completes the proof of Theorem 3.

In the course of the proof of Theorem 3, we obtain the

Corollary. *If R is a semiprime ring with a derivation d such that $ax+d(x) \in G$ and $ad(x) + d^2(x) \in I$ for all x in R, then $ad + d^2 = 2d(R)^2 = 0$.*

A simple calculation shows that $d(x, y, z)) = (d(x), y, z) + (x, d(y), z) + (x, y, d(z))$ for all x, y, z in R. Applying this we have that all the above associative subrings of R are d-invariant. Thus $d(G) \subseteq G$. Hence we can give another proof of Theorem 3.

Another proof of Theorem 3. For all x, y in R, we have

$$a(xy) + d(xy) = x(ay + d(y)) + d(x)y = (ax + d(x))y + xd(y) \in G.$$

Then with $x \in G$, or $y \in G$ in this equality respectively, and using the hypothesis, and noting that G is a subring of R, we obtain $d(x)y \in G$ and $xd(y) \in G$. Thus, we get $d(G)R + Rd(G) \subseteq G$. Hence by Lemma 1, the ideal H of R generated by $d(G)$ is

$$H = \sum d(G) + d(G)R + Rd(G) + R \cdot d(G)R.$$

Applying $d(G) \subseteq G$, (2), (8) and $d(G)R + Rd(G) \subseteq G$, we have $d(G) \cdot I = 0$. Thus, we get $H \cdot I = 0$. By the primeness of R, this implies $H = 0$ or $I = 0$. If $I = 0$, then R is associative. Assume that $H = 0$. Then $d(G) = 0$ and so $(ad + d^2)(x) = ad(x) + d^2(x) = d(ax + d(x)) = 0$ for all x in R. Hence, R is associative or $ad + d^2 = 0$.

Finally, we pose the following more general

Problem. If R is a prime ring with a derivation d and there exists a polynomial $f(t) \in \mathbf{Z}[t]$ such that $f(d)(x) \in G$ for all x in R, then either R is associative or $g(d) = 0$ for some $g(t) \in \mathbf{Z}[t]$.

By Theorems A, B and 3, we know that this problem is true when $f(t) = t$, t^n, $a+t$, and $g(t) = 2t$, t^{3n-1} and $(a + t)t$ respectively.

For the problem, we have the following remark by using the standard argument in linear algebra.

Remark. If R is a nonassociative algebra over a field F and d is a derivation of R such that $f(d)(R) \subseteq G$, where

$$f(t) = \Pi_{i=1}^{n}(a_i + t)^{m_i} \in F[t], a_i \neq a_j \qquad \text{if } i \neq j,\ i, j \in \{1, 2, \cdots, n\},$$

then $R = \sum_{i=1}^{n} R_i$ is a vector subspaces sum, where

$$R_i = \{x \in R : (a_i + d)^{m_i}(x) \in G\},$$

and

$$R_i \cap \sum_{j \neq i_{j=1}}^{n} R_j = G, \qquad i = 1, 2, \cdots, n.$$

Moreover, if $m_i = 1$ and $GR_i + R_iG \subseteq R_i$, $i = 1, 2, \cdots, n$, then as another proof of Theorem 3 we have

$$d(G)R_i + R_i d(G) \subseteq G, i = 1, 2, \cdots, n,$$

and so

$$d(G)R + Rd(G) \subseteq G.$$

Thus, if R is prime then as above we can prove that R is associative or $d(G) = 0$ which implies $f(d)d = 0$. For the general case, if $char(R) = 0$ then we may consider the tensor algebra $R \otimes_F E$, where E is the algebraic closure of F.

We note that Theorem 3 can be generalized to skew derivation. Some lemmas and theorems of [6] have extended to s-derivations d with $sd = ds$. The above results will be a part of my doctoral thesis at Taiwan University under the guidance of Professor Pjek-Hwee Lee. I thank him very much and I also thank my teacher Professor Tsiu-Kwen Lee. Finally, the author thanks the referee for careful comments.

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