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ON CERTAIN GENERALIZATIONS OF THE SPIRAL-LIKE AND ROBERTSON FUNCTIONS

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Abstract. Let $S^{\lambda}(\alpha,\beta,A,B)$ denote the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the unit disc $U=\{z:|z|<1\}$ and satisfy the inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{(B-A)\beta(\frac{zf'(z)}{f(z)}-1+(1-\alpha)e^{-i\lambda}\cos\lambda)+A(\frac{zf'(z)}{f(z)}-1)}\right|<1$$

for some λ, α, β, A, $B\,(|\lambda|<\pi/2,\ 0\le\alpha<1,\ 0<\beta\le1,\ -1\le A<B\le1$ and $0<B\le1)$ and for all $z\in U$. Further $f(z)$ is said to belong to the class $C^{\lambda}(\alpha,\beta,A,B)$ $(|\lambda|<\pi/2, 0\le\alpha<1, 0<\beta\le1, -1\le A<B\le1$ and $0<B\le1)$ if and only if $zf'(z)\in S^{\lambda}(\alpha,\beta,A,B)$. In the present paper, the authors give several representation formulas, distortion theorems, and coefficient bounds for functons belonging to these classes. They also obtain the sharp radius of γ-spiral and starlikeness for the class $S^{\lambda}(\alpha,\beta,A,B)$ and the sharp radius of γ-convex and convexity for the class $C^{\lambda}(\alpha,\beta,A,B)$.

1. Introduction

Let S denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic in the unit disc $U=\{z:|z|<1\}$. We use Ω to denote the class of bounded analytic functions $w(z)$ in U, satisfying the conditions $w(0)=0$ and $|w(z)|<1$ for $z\in U$.

For $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<1$ and $-1\le A<B\le1, 0<B\le1$, let $S^{\lambda}(\alpha,A,B)$ be the class of those functions $f(z)$ of S for which $\frac{zf'(z)}{f(z)}$ is subordinate to $\frac{1+[B+(A-B)(1-\alpha)e^{-i\lambda}\cos\lambda]z}{1+Bz}$.

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In other words $f(z) \in S^\lambda(\alpha, A, B)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1 + [B + (A - B)(1-\alpha)e^{-i\lambda}\cos\lambda]w(z)}{1 + B\,w(z)}. \tag{1.2}$$

And the above condition is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)(\frac{zf'(z)}{f(z)} - 1 + (1-\alpha)e^{-i\lambda}\cos\lambda) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < 1, \ z \in U. \tag{1.3}$$

The class $S^\lambda(\alpha, A, B)$ was introduced by Aouf [4].

Motivated by [1,14,23], we in the present paper, introduce the classes $S^\lambda(\alpha, \beta, A, B)$ and $C^\lambda(\alpha, \beta, A, B)$, defined as follows:

Definition 1. A function $f(z) \in S$ is in the class $S^\lambda(\alpha, \beta, A, B)$ if and only if the inequality

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)\beta(\frac{zf'(z)}{f(z)} - 1 + (1-\alpha)e^{-i\lambda}\cos\lambda) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < 1 \tag{1.4}$$

holds for some λ, α, β, A and B ($|\lambda| < \frac{\pi}{2}$; $0 \le \alpha < 1$; $0 < \beta \le 1$; $-1 \le A < B \le 1$; $0 < B \le 1$), and for all $z \in U$.

Definition 2. A function $f(z) \in S$ is in the class $C^\lambda(\alpha, \beta, A, B)$ if and only if the inequality

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{(B-A)\beta(\frac{zf''(z)}{f'(z)} + (1-\alpha)e^{-i\lambda}\cos\lambda) + A\frac{zf''(z)}{f'(z)}} \right| < 1 \tag{1.5}$$

holds for some λ, α, β, A and B ($|\lambda| < \frac{\pi}{2}$; $0 \le \alpha < 1$; $0 < \beta \le 1$; $-1 \le A < B \le 1$; $0 < B \le 1$), and for all $z \in U$.

It follows immediately from Definition 1 and Definition 2 that

$$f(z) \in C^\lambda(\alpha, \beta, A, B) \text{ if and only if } zf'(z) \in S^\lambda(\alpha, \beta, A, B). \tag{1.6}$$

We note that $S^\lambda(\alpha, \beta, -1, 1) = S^\lambda(\alpha, \beta)$ is the class of λ-spiral-like functions of order α and type β ($|\lambda| < \frac{\pi}{2}$; $0 \le \alpha < 1$; $0 < \beta \le 1$) which was studied earlier by Mogra and Ahuja [23]. On the other hand, $C^\lambda(\alpha, \beta, -1, 1) = C^\lambda(\alpha, \beta)$ is the class of λ-Robertson functions of order α and type β ($|\lambda| < \frac{\pi}{2}$; $0 \le \alpha < 1$; $0 < \beta \le 1$) which was studied earlier by Ahuja [1].

Furthermore, by specializing the parameters λ, α, β, A and B, we obtain the following subclasses studied earlier by various authors.

(i) $S^o(\alpha, \beta, A, B) = S^*(\alpha, \beta, A, B)$ (Aouf [6]);

(ii) $S^o(\alpha, 1, A, B) = S^*(\alpha, , A, B)$ (Aouf [5]);
(iii) $S^o(0, 1, A, B) = S^*(A, B)$ (Goel and Mehrok [10,11], Janowski [13]);
(iv) $S^\lambda(0, 1, A, B) = S^\lambda(A, B)$ (Dashrath and Shukla [9], Kumar and Shulka [17]);
(v) $S^\lambda(\alpha, 1, A, B) = S^\lambda(\alpha, A, B)$ (Aouf [2]);
(vi) $S^\lambda(\alpha, 1, -1, 1) = S^\lambda(\alpha)$ (Libera [18], and Patil and Thakare [26]);
(vii) $S^\lambda(0, 1, -1, 1) = S^\lambda$ (Špaček [29] and Zamorski [31]);
(viii) $S^\lambda(0, \frac{2-\cos\lambda}{2}, -1, 1) = H(\lambda)$ (Goel [10]);
(ix) $S^\lambda(\frac{1-\beta+2\alpha\beta}{1+\beta}, \frac{1+\beta}{2}, -1, 1) = S^\lambda_{\alpha,\beta}$ (Makówka [20] and Gopalakrishna and Umarani [12]);
(x) $S^o(\frac{1-\beta}{1+\beta}, \frac{1+\beta}{2}, -1, 1) = S(\beta)$ (Padmanabhan [25] and Mogra [21]);
(xi) $S^o(\alpha, \beta, -1, 1) = S^*(\alpha, \beta)$ (Juneja and Mogra [14]);
(xii) $S^o(\alpha, \frac{1}{2}, -1, 1) = \overline{S}_\alpha$ (Wright [30]);
(xiii) $C^\lambda(\alpha, 1, -1, 1) = C^\lambda(\alpha)$ (Chichra [8] and Sizuk [28]);
(xiv) $C^\lambda(1, 1, -1, 1) = C^\lambda$ (Robertson [27], Libera and Ziegler [19], and Bajpai and Mehrok [7]);
(xv) $S^\lambda(0, \frac{2M-1}{2M}, -1, 1) = F_{\lambda,M}$ and $C^\lambda(0, \frac{2M-1}{2M}, -1, 1) = G_{\lambda,M} (M > \frac{1}{2})$ (Kulshresha [16]),

and

(xvi) $S^\lambda(\alpha, \frac{2M-1}{2M}, -1, 1) = F_M(\lambda, \alpha)$ and $C^\lambda(\alpha, \frac{2M-1}{2M}, -1, 1) = G_M(\lambda, \alpha)\, (M > \frac{1}{2})$ (Aouf [2,3]).

Note. Although $S^\lambda(\alpha, \beta, A, B) \subset S^\lambda(\alpha, \beta)$ and $C^\lambda(\alpha, \beta, A, B) \subset C^\lambda(\alpha, \beta)$, the functions in the class $C^\lambda(\alpha, \beta, A, B)$ need not be univalent in U, as shown in [1,27].

2. Representation Formulas

Let Q denote the class of functions $\varphi(z)$ which are analytic in U and which satisfy $|\varphi(z)| \le 1$ for all $z \in U$. We require the following lemma.

Lemma 1. *If a function*

$$H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n. \tag{2.1}$$

is analytic in U and satisfies the condition

$$\left| \frac{H(z) - 1}{(B - A)\beta(H(z) - 1 + (1 - \alpha)e^{-i\lambda} \cos\lambda) + A(H(z) - 1)} \right| < 1, \tag{2.2}$$

for $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$, and for all $z \in U$, then

$$H(z) = \frac{1 + \{[(B - A)\beta + A] - (B - A)\beta(1 - \alpha)e^{-i\lambda} \cos\lambda\} z\varphi(z)}{1 + [(B - A)\beta + A]\, z\,\varphi(z)} \tag{2.3}$$

for some $\varphi(z) \in Q$. *Conversely, a function* $H(z)$ *given by* (2.2) *for some* $\varphi(z) \in Q$ *is analytic in* U *and satisfies* (2.2) *for all* $z \in U$.

Proof. The first part of Lemma 1 is obtained immediately by an application of Schwarz's Lemma [24]. The second part follows from the observation that the function

$$w = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}z}{1 + [(B-A)\beta + A]\, z} \tag{2.4}$$

maps U onto the disc

$$\left| \frac{1-w}{(B-A)\beta(w - 1 + (1-\alpha)e^{-i\lambda}\cos\lambda) + A(w-1)} \right| < 1 \tag{2.5}$$

in the w-plane.

Theorem 1. *A function* $f(z)$, *defined by* (1.1) *and analytic in* U, *is in the class* $S^{\lambda}(\alpha, \beta, A, B)$ *if and only if*

$$f(z) = z \exp\left\{ -(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda \int_0^z \frac{\varphi(t)}{1 + [(B-A)\beta + A]t\varphi(t)}\,dt \right\}. \quad (z \in U) \tag{2.6}$$

for some $\varphi(z) \in Q$.

Proof. First suppose that $f(z) \in S^{\lambda}(\alpha, \beta, A, B)$. Noting that $\frac{zf'(z)}{f(z)}$ satisfies the hypothesis of the first part of Lemma 1, we see that

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}z\varphi(z)}{1 + [(B-A)\beta + A]\, z\varphi(z)} \tag{2.7}$$

for some function $\varphi(z) \in Q$. It is easily observed from (2.7) that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = -\frac{(B-A)\beta(1-\alpha)e^{-i\lambda}\varphi(z)\cos\lambda}{1 + [(B-A)\beta + A]\, z\varphi(z)}. \tag{2.8}$$

Upon integrating both sides of (2.8) from 0 to z, if we exponentiate the resulting equation, we obtain the representation formula (2.6).

Conversely, if (2.6) holds true, then

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}z\varphi(z)}{1 + [(B-A)\beta + A]\, z\varphi(z)} \quad (z \in U;\ \varphi(z) \in Q). \tag{2.9}$$

Now Theorem 1 follows by appealing to the second part of Lemma 1.

An immediate consequence of Theorem 1, and a representation theorem for functions in $S^*(\alpha,\beta,A,B)$ given by Aouf [6] may be shown in the following corollary:

Corollary 1. *Let the function $f(z)$ defined by (1.1). Then $f(z) \in S^\lambda(\alpha,\beta,A,B)$ if and only if there is a function $f_1(z) \in S^*(\alpha,\beta,A,B)$ such that*

$$f(z) = z\left[\frac{f_1(z)}{z}\right]^{e^{-i\lambda}\cos\lambda} \quad (z \in U). \tag{2.10}$$

In view of the relationship (1.6), it is not difficult to deduce from the above results the following representation formulas for functions belonging to the class $C^\lambda(\alpha,\beta,A,B)$:

Corollary 2. *A function $f(z)$ defined by (1.1) is in the class $C^\lambda(\alpha,\beta,A,B)$ if and only if its derivative $f'(z)$ can be represented as follows:*

(i)

$$f'(z) = [f_2'(z)]^{e^{-i\lambda}\cos\lambda} \tag{2.11}$$

for $f_2(z) \in C^o(\alpha,\beta,A,B) = C^*(\alpha,\beta,A,B)$;

(ii)

$$f'(z) = \exp\left\{-(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda \int_0^z \frac{\varphi(t)\,dt}{1+[(B-A)\beta+A]t\,\varphi(t)}\right\}, \tag{2.12}$$

for some function $\varphi(z) \in Q$.

3. A Sufficient Condition

We now establish a sufficient condition for a function to be in each of the classes $S^\lambda(\alpha,\beta,A,B)$ and $C^\lambda(\alpha,\beta,A,B)$.

Theorem 2. *Let the function $f(z)$ defined by (1.1) be analytic in U. Then $f(z) \in S^\lambda(\alpha,\beta,A,B)$ if, for some λ, α, A and B ($|\lambda| < \frac{\pi}{2}$; $0 \le \alpha < 1$; $-1 \le A < B \le 1$; $0 < B \le 1$),*

$$\sum_{n=2}^{\infty}\left\{n[1-A-(B-A)\beta]-1+\left|[-A-(B-A)\beta]+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\right\}|a_n| \le (B-A)\beta(1-\alpha)\cos\lambda, \tag{3.1}$$

whenever $0 < \beta \le (\frac{A}{A-B})$, and

$$\sum_{n=2}^{\infty}\left\{(n-1)+\left|[(B-A)\beta+A](n-1)+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\right\}|a_n| \le (B-A)\beta(1-\alpha)\cos\lambda, \tag{3.2}$$

whenever $(\frac{A}{A-B}) \leq \beta \leq 1$.

Proof. Let $|z| = r < 1$. Noting that

$$|zf'(z) - f(z)| < \sum_{n=2}^{\infty}(n-1)\,|a_n|\,r, \tag{3.3}$$

and

$$\begin{aligned}&\left|(B-A)\beta[zf'(z) - f(z) + (1-\alpha)e^{-i\lambda} f(z)\cos\lambda] + A[zf'(z) - f(z)]\right| \\ &\quad > \left\{(B-A)\beta(1-\alpha)\cos\lambda - \sum_{n=2}^{\infty}[-A-(B-A)\beta]n\,|a_n| \right.\\ &\qquad \left. - \sum_{n=2}^{\infty}\left|[-A-(B-A)\beta] + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\,|a_n|\right\} r,\end{aligned} \tag{3.4}$$

we see that

$$\begin{aligned}&|zf'(z) - f(z)| - |(B-A)\beta[zf'(z) - f(z) + (1-\alpha)e^{-i\lambda} f(z)\cos\lambda] \\ &+A[zf'(z)-f(z)]| < \left[\sum_{n=2}^{\infty}\{n[1-A-(B-A)\beta - 1 + |[-A-(B-A)\beta]\right. \\ &\left. + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}\,|a_n| - (B-A)\beta(1-\alpha)\cos\lambda\right] r,\end{aligned} \tag{3.5}$$

provided that $0 < \beta \leq (\frac{A}{A-B})$. The right-hand side of (3.5) is non-positive by (3.1), so that $f(z) \in S^{\lambda}(\alpha, \beta, A, B)$ by Definition 1.

For the second part, we assume that (3.2) holds true for $(\frac{A}{A-B}) \leq \beta \leq 1$. In this case, we observe that

$$\begin{aligned}&|(B-A)\beta[zf'(z) - f(z) + (1-\alpha)e^{-i\lambda} f(z)\cos\lambda] + A[zf'(z) - f(z)]| \\ &>\left\{(B-A)\beta(1-\alpha)\cos\lambda - \sum_{n=2}^{\infty}\left|[(B-A)\beta + A](n-1)\right.\right. \\ &\quad \left.\left. + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\,|a_n|\right\} r.\end{aligned} \tag{3.6}$$

Making using of (3.3), (3.6), and (3.2), we complete the proof of Theorem 2.

Corollary 3. *Let the function* $f(z)$ *defined by* (1.1) *be analytic in* U. *Then* $f(z)$ *is in the class* $C^{\lambda}(\alpha, \beta, A, B)$ *if, for some* λ, α, A *and* B $(|\lambda| < \frac{\pi}{2};\ 0 \leq \alpha < 1;\ -1 \leq A < B \leq 1;\ 0 < B \leq 1)$.

$$\begin{aligned}&\sum_{n=2}^{\infty} n\left\{n[1-A-(B-A)\beta] - 1 + \left|[-A-(B-A)\beta] + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\right\}|a_n| \\ &\qquad \leq (B-A)\beta(1-\alpha)\cos\lambda,\end{aligned} \tag{3.7}$$

whenever $0 < \beta \le (\frac{A}{A-B})$, *and*

$$\sum_{n=2}^{\infty} n\left\{(n-1) + \left|[(B-A)\beta + A](n-1) + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|\right\} |a_n| \le (B-A)\beta(1-\alpha)\cos\lambda, \tag{3.8}$$

whenever $(\frac{A}{A-B}) \le \beta \le 1$.

Proof. Since

$$zf'(z) = z + \sum_{n=2}^{\infty} n\, a_n\, z^n, \tag{3.9}$$

by replacing a_n by $n\, a_n$ in Theorem 2, we immediately have Corollary 3 in view of the equivalence relation (1.6).

For various choices of the parameters involved in Theorem 2 and Corollary 3, we can obtain the corresponding results for functions belonging to the numerous simpler classes described in Section 1.

4. Ditortion Theorems

Theorem 3. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $S^{\lambda}(\alpha,\beta,A,B)$, *then*

$$|f(z)| \le r\left[\frac{(1+[(B-A)\beta+A]r)^{(1-\cos\lambda)}}{(1-[(B-A)\beta+A]r)^{(1+\cos\lambda)}}\right]^{\frac{(B-A)\beta(1-\alpha)\cos\lambda}{2[(B-A)\beta+A]}}, \tag{4.1}$$

and

$$|f(z)| \ge r\left[\frac{(1-[(B-A)\beta+A]r)^{(1-\cos\lambda)}}{(1+[(B-A)\beta+A]r)^{(1+\cos\lambda)}}\right]^{\frac{(B-A)\beta(1-\alpha)\cos\lambda}{2[(B-A)\beta+A]}}, \tag{4.2}$$

for $|z| = r\,(0 < r < 1)$, $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $0 < \beta \le 1$, $\beta \ne (\frac{A}{A-B})$, $-1 \le A < B \le 1$, *and* $0 < B \le 1$; *and*

$$|f(z)| \le r\,\exp[-A(1-\alpha)r\cos\lambda] \tag{4.3}$$

and

$$|f(z)| \ge r\,\exp[A(1-\alpha)r\cos\lambda] \tag{4.4}$$

for $|z| = r\,(0 < r < 1)$, $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $\beta = (\frac{A}{A-B})$, $-1 \le A < B \le 1$, *and* $0 < B \le 1$.

All these estimates are sharp for all admissible values of λ, α, β, A *and* B.

Proof. Since $f(z) \in S^{\lambda}(\alpha,\beta,A,B)$, the condition (1.4) coupled with an application of Schwarz's Lemma [24] implies that

$$\left|\frac{zf'(z)}{f(z)} - \xi\right| < R,$$

where

$$\xi = \left[1 - [(B-A)\beta + A]\left\{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)\cos^2\lambda\right\} r^2 \right.$$
$$\left. -i2^{-1}(B-A)\beta[(B-A)\beta + A](1-\alpha)r^2 \sin 2\lambda\right] \left[1 - [(B-A)\beta + A]^2 r^2\right]^{-1}$$

and

$$R = \frac{(B-A)\beta(1-\alpha)r\,\cos\lambda}{1 - [(B-A)\beta + A]^2 r^2} \quad (|z| = r).$$

Hence we obtain

$$\frac{1 - (B-A)\beta(1-\alpha)r\cos\lambda + [(B-A)\beta + A]\left\{(B-A)\beta(1-\alpha)\cos^2\lambda - [(B-A)\beta + A]\right\} r^2}{1 - [(B-A)\beta + A]^2 r^2}$$
$$\le \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \le$$
$$\frac{1 + (B-A)\beta(1-\alpha)r\cos\lambda + [(B-A)\beta + A]\left\{(B-A)\beta(1-\alpha)\cos^2\lambda - [(B-A)\beta + A]\right\} r^2}{1 - [(B-A)\beta + A]^2 r^2}. \quad (4.5)$$

Observing that

$$\log\left(\left|\frac{f(z)}{z}\right|\right) = \operatorname{Re}\left(\log\frac{f(z)}{z}\right) = \operatorname{Re}\left[\int_0^z \left(\frac{f'(s)}{f(s)} - \frac{1}{s}\right) ds\right]$$
$$= \int_0^r \frac{1}{t}\operatorname{Re}\left[te^{i\theta}\frac{f'(t\,e^{i\theta})}{f(t\,e^{i\theta})} - 1\right] dt,$$

and applying (4.5), we find that

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \le [(B-A)\beta(1-\alpha)\,\cos\lambda] \int_0^r \frac{1 + [(B-A)\beta + A]t\,\cos\lambda}{1 - [(B-A)\beta + A]^2 t^2}\, dt. \quad (4.6)$$

Now suppose that $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $\beta \ne (\frac{A}{A-B})$, $-1 \le A < B \le 1$, and $0 < B \le 1$. Then (4.6) yields

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2[(B-A)\beta + A]} \log\left\{\frac{(1 + [(B-A)\beta + A]r)^{(1-\cos\lambda)}}{(1 - [(B-A)\beta + A]r)^{(1+\cos\lambda)}}\right\},$$

which leads us to (4.1). For the case when $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $\beta = (\frac{A}{A-B})$, $-1 \le A < B \le 1$, and $0 < B \le 1$, (4.6) immediately gives (4.3).

In view of the fact that

$$\log\left(\left|\frac{f(z)}{z}\right|\right) = \operatorname{Re}\left(\log\frac{f(z)}{z}\right) = \int_0^r \operatorname{Re}\left\{\frac{\partial}{\partial t}\left[\log\frac{f(t)}{t}\right]\right\} dt$$
$$= \int_0^r \frac{1}{t}\operatorname{Re}\left\{\frac{tf'(t)}{f(t)} - 1\right\} dt,$$

and with the aid of (4.5), we may write

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \geq -[(B-A)\beta(1-\alpha)\cos\lambda]\int_0^r \frac{1-[(B-A)\beta+A]t\cos\lambda}{1-[(B-A)\beta+A]^2t^2}\,dt. \quad (4.7)$$

If $\beta \neq (\frac{A}{A-B})$, then (4.2) follows upon evaluating the integeral in (4.7). If, on the other hand, $\beta = (\frac{A}{A-B})$, then we immediately get (4.4) from (4.7).

The extermal function for all of the inequalities is given by

$$f(z) = \begin{cases} z\left\{1-[(B-A)\beta+A]ze^{i\theta}\right\}^{\frac{-[(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A]}}, & \beta \neq (\frac{A}{A-B}) \\ z\exp\left[-A(1-\alpha)ze^{i(\theta-\lambda)}\cos\lambda\right], & \beta = (\frac{A}{A-B}) \end{cases} \quad (4.8)$$

where $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and $\theta(0 \leq \theta \leq 2\pi)$ is determined by

$$\tan(\frac{\theta}{2}) = \left\{\frac{1-[(B-A)\beta+A]r}{1+[(B-A)\beta+A]r}\right\}\cot(\frac{\pi}{2}-\frac{\lambda}{2}) \quad (4.9)$$

for the equality in (4.1) and (4.3) and by the equation

$$\tan(\frac{\theta}{2}) = \left\{\frac{1-[(B-A)\beta+A]r}{1+[(B-A)\beta+A]r}\right\}\cot(-\frac{\lambda}{2}) \quad (4.10)$$

for the equality in (4.2) and (4.4).

Corollary 4. *If a function $f(z)$ defined by (1.1) is in the class $C^\lambda(\alpha,\beta,A,B)$, then*

$$|f'(z)| \leq \left[\frac{(1+[(B-A)\beta+A]r)^{(1-\cos\lambda)}}{1-[(B-A)\beta+A]r)^{(1+\cos\lambda)}}\right]^{\frac{(B-A)\beta(1-\alpha)\cos\lambda}{2[(B-A)\beta+A]}} \quad (4.11)$$

and

$$|f'(z)| \geq \left[\frac{(1-[(B-A)\beta+A]r)^{(1-\cos\lambda)}}{1+[(B-A)\beta+A]r)^{(1+\cos\lambda)}}\right]^{\frac{(B-A)\beta(1-\alpha)\cos\lambda}{2[(B-A)\beta+A]}} \quad (4.12)$$

for $|z| = r < 1$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\beta \neq (\frac{A}{A-B})$, $-1 \leq A < B \leq 1$, *and* $0 < B \leq 1$; *and*

$$|f'(z)| \leq \exp[-A(1-\alpha)r\cos\lambda] \quad (4.13)$$

and

$$|f'(z)| \geq \exp[A(1-\alpha)r\cos\lambda], \quad (4.14)$$

for $|z| = r < 1$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $\beta = (\frac{A}{A-B})$, $-1 \leq A < B \leq 1$, *and* $0 < B \leq 1$.

The function $f(z)$ given by

$$f'(z) = \begin{cases} \left\{1-[(B-A)\beta+A]ze^{i\theta}\right\}^{\frac{-[(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A]}}, & \beta \neq (\frac{A}{A-B}) \\ \exp\left[-A(1-\alpha)\,z\,e^{i(\theta-\lambda)}\cos\lambda\right], & \beta = (\frac{A}{A-B}) \end{cases} \quad (4.15)$$

provides equality in (4.11) *and* (4.13) *when* θ *is given by Equation* (4.9). *The function* $f(z)$ *given by* (4.15) *also provides equality in* (4.12) *and* (4.14) *when* θ *is given by Equation* (4.10).

For various choices of the parameters in Theorem 3 and Corollary 4, the corresponding known or new results can be deduced for functions in the classes studied earlieir in the literature.

5. Coefficient Bounds

We shall require the following two lemmas in our investigation

Lemma 2 [24]. *Let the function* $w(z)$ *defined by*

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \tag{5.1}$$

be in the class Ω. *Then*

$$|c_1| \leq 1 \tag{5.2}$$

and

$$|c_2| \leq 1 - |c_1|^2 \tag{5.3}$$

Lemma 3 [15]. *Let the function* $w(z)$ *defined by* (5.1) *be in the class* Ω. *Then*

$$|c_2 - \nu c_1^2| \leq \max\{1, |\nu|\} \tag{5.4}$$

for any complex number ν. *Equality in* (5.4) *may be attained with the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\nu| < 1$ *and* $|\nu| \geq 1$, *respectively.*

Theorem 4. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $S^{\lambda}(\alpha, \beta, A, B)$, $\beta \neq (\frac{A}{A-B})$, *then*

(a) *for any real number* μ, *we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2}\left[\cos\lambda\left\{(B-A)\beta(1-\alpha)(1-2\mu)\right.\right. \\ \quad \left.\left. + [(B-A)\beta + A]\right\} + |[(B-A)\beta + A]\sin\lambda|\right], \\ \qquad \text{if } \mu \leq \frac{-1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)}, \\ \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2}\left[\cos\lambda + |[(B-A)\beta + A]\sin\lambda|\right], \\ \quad \text{if } \frac{-1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)} \leq \mu \leq \frac{1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)}, \\ \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2}\left[\cos\lambda\left\{(B-A)\beta(1-\alpha)(2\mu-1)\right.\right. \\ \quad \left.\left. - [(B-A)\beta + A]\right\} + |[(B-A)\beta + A]\sin\lambda|\right], \\ \qquad \text{if } \mu \geq \frac{1+A+(B-A)\beta(2-\alpha)}{2(B-A)\beta(1-\alpha)}, \end{cases} \tag{5.5}$$

and

(b) *for any complex number* μ, *we obtain*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \max\{1, |(B-A)\beta(1-\alpha)(2\mu-1)\cos\lambda - [(B-A)\beta + A]\, e^{i\lambda}|\}. \tag{5.6}$$

This result is sharp for each μ either real or complex.

Proof. Since $f(z) \in S^\lambda(\alpha, \beta, A, B)$, (2.7) gives

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1 + [(B-A)\beta + A]w(z)}, \tag{5.7}$$

where $w(z)$ defined by (5.1) is in the class Ω. Rewriting (5.7) in the form

$$w(z) = \frac{zf'(z) - f(z)}{\{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}f(z) - [(B-A)\beta + A]zf'(z)} \tag{5.8}$$

and applying the definition (1.1), it can be shown that

$$w(z) = -\frac{1}{(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda} \cdot \left[a_2 z + \left\{2a_3 - \frac{[(B-A)\beta + A] + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda} a_2^2\right\} z^2 + \cdots\right]. \tag{5.9}$$

Now compare the coefficients of z and z^2 on both sides of (5.9), using the definition (5.1). We thus obtain

$$c_1 = -\frac{e^{i\lambda}\sec\lambda}{(B-A)\beta(1-\alpha)} a_2 \tag{5.10}$$

and

$$c_2 = -\frac{2e^{i\lambda}\sec\lambda}{(B-A)\beta(1-\alpha)} a_3 + \frac{e^{i\lambda}\sec\lambda}{(B-A)^2\beta^2(1-\alpha)^2}\{(B-A)\beta(1-\alpha) + [(B-A)\beta + A]e^{i\lambda}\sec\lambda\}\, a_2^2. \tag{5.11}$$

Consequently, we have

$$a_2 = -\frac{(B-A)\beta(1-\alpha)}{\sec\lambda} c_1\, e^{-i\lambda} \tag{5.12}$$

and

$$a_3 = -\frac{(B-A)\beta(1-\alpha)}{2\sec\lambda} c_2\, e^{-i\lambda} + \left[\frac{(B-A)\beta(1-\alpha) + [(B-A)\beta + A]e^{i\lambda}\sec\lambda}{2(B-A)\beta(1-\alpha)}\right] a_2^2. \tag{5.13}$$

Using (5.12) and (5.13), we get

$$a_3 - \mu a_2^2 = -\frac{(B-A)\beta(1-\alpha)}{2e^{i\lambda}\sec\lambda} c_2$$
$$+ \left[\frac{(B-A)\beta(1-\alpha) + [(B-A)\beta + A]e^{i\lambda}\sec\lambda}{2(B-A)\beta(1-\alpha)} - \mu\right] \frac{(B-A)^2\beta^2(1-\alpha)^2}{e^{2i\lambda}\sec^2\lambda} c_1^2. \quad (5.14)$$

Thus taking modulus of both sides of (5.14), we are led to

$$|a_3 - \mu a_2^2| = \frac{(B-A)\beta(1-\alpha)}{2\sec\lambda} \cdot$$
$$\cdot \left| c_2 - \left\{ -\frac{(B-A)\beta(1-\alpha)}{e^{i\lambda}\sec\lambda}(2\mu-1) + [(B-A)\beta + A] \right\} c_1^2 \right|. \quad (5.15)$$

(a) When μ is real.
For real μ, (5.15) becomes

$$|a_3 - \mu a_2^2| \le \frac{(B-A)\beta(1-\alpha)}{2\sec\lambda} \cdot$$
$$\cdot \left\{ |c_2| + \left| (B-A)\beta(1-\alpha)(2\mu-1)\cos\lambda - [(B-A)\beta + A]e^{i\lambda} \right| |c_1|^2 \right\}. \quad (5.16)$$

Applying Lemma 2 for $|c_2|$ in (5.16) we obtain

$$|a_3 - \mu a_2^2| \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \{1 + [|(B-A)\beta(1-\alpha)(2\mu-1)\cos\lambda$$
$$-[(B-A)\beta + A]e^{i\lambda}| - 1] \; |c_1|^2\}. \quad (5.17)$$

Again using Lemma 2 for $|c_1|$ in (5.17) we are led to

$$|a_3 - \mu a_2^2| \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \{|(B-A)\beta(1-\alpha)(2\mu-1)\cos\lambda$$
$$-[(B-A)\beta + A]e^{i\lambda}|\}$$
$$\le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \{\cos\lambda \, |(B-A)\beta(1-\alpha)(2\mu-1)$$
$$-[(B-A)\beta + A]| + |[(B-A)\beta + A]\sin\lambda|\}. \quad (5.18)$$

Thus from (5.18) with simple computations we obtain the results of (5.5) stated in (a) of Theorem 4 for various values of real μ.

(b) When μ is a Complex Number.
For any complex number μ (5.15) may be written as

$$|a_3 - \mu a_2^2| \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \cdot$$
$$\cdot \left| c_2 - \left\{ \frac{[(B-A)\beta + A]e^{i\lambda}\sec\lambda - (B-A)\beta(1-\alpha)(2\mu-1)}{e^{i\lambda}\sec\lambda} \right\} c_1^2 \right|. \quad (5.19)$$

Applying Lemma 3 in (5.19) we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(B-A)\beta(1-\alpha)\cos\lambda}{2} \max\{1,$$
$$\left|(B-A)\beta(1-\alpha)(2\mu-1)\cos\lambda - [(B-A)\beta + A]e^{i\lambda}\right|\} \tag{5.20}$$

which is (5.6) in (b) of Theorem 4.

Finally, the assertions (5.5) and (5.6) of Theorem 4 are sharp in view of the fact that the assertion (5.4) of Lemma 3 is sharp.

In a similar way we can prove:

Theorem 5. *If a function $f(z)$ defined by* (1.1) *is in the class $C^\lambda(\alpha,\beta,A,B)$, $\beta \neq (\frac{A}{A-B})$, then*

(a) *for any real number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{(B-A)\beta(1-\alpha)\cos\lambda}{6}\left[\cos\lambda\left\{(B-A)\beta(1-\alpha)(1-\frac{3}{2}\mu)\right.\right. \\ \left.\left. + [(B-A)\beta + A]\right\} + \left|[(B-A)\beta + A]\sin\lambda\right|\right], \\ \quad \textit{if } \mu \leq \frac{2[-1+A+(B-A)\beta(2-\alpha)]}{3(B-A)\beta(1-\alpha)}, \\ \frac{(B-A)\beta(1-\alpha)\cos\lambda}{6}\left[\cos\lambda + \left|[(B-A)\beta + A]\sin\lambda\right|\right], \\ \textit{if } \frac{2[-1+A+(B-A)\beta(2-\alpha)]}{3(B-A)\beta(1-\alpha)} \leq \mu \leq \frac{2[1+A+(B-A)\beta(2-\alpha)]}{3(B-A)\beta(1-\alpha)}, \\ \frac{(B-A)\beta(1-\alpha)\cos\lambda}{6}\left[\cos\lambda\left\{(B-A)\beta(1-\alpha)(\frac{3}{2}\mu-1)\right.\right. \\ \left.\left. - [(B-A)\beta + A]\right\} + \left|[(B-A)\beta + A]\sin\lambda\right|\right], \\ \textit{if } \mu \geq \frac{2[1+A+(B-A)\beta(2-\alpha)]}{3(B-A)\beta(1-\alpha)}, \end{cases}$$

and

(b) *for any complex number μ, we obtain*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(B-A)\beta(1-\alpha)\cos\lambda}{6} \max\left\{1, \left|(B-A)\beta(1-\alpha)(\frac{3}{2}\mu - 1)\cos\lambda\right.\right.$$
$$\left.\left. - [(B-A)\beta + A]\, e^{i\lambda}\right|\right\}.$$

This result is sharp for each μ either real or complex.

Theorem 6. *Let the function $f(z)$ defined by* (1.1) *be in the class $S^\lambda(\alpha,\beta,A,B)$.*

(a) *If*

$$\beta(1-\alpha)(k-\alpha)\cos^2\lambda > \frac{(k-1)}{(B-A)^2\beta}\left\{(k-1)(1-A^2) - (B-A)\beta[(B-A)\beta + 2A]\cdot\right.$$
$$\left.\cdot[(1-\alpha)\cos^2\lambda + k - 1]\right\}, \tag{5.21}$$

let

$$N = \left[\frac{\beta(1-\alpha)(k-\alpha)\cos^2\lambda}{\frac{(k-1)}{(B-A)^2\beta}\{(k-1)(1-A^2) - (B-A)\beta[(B-A)\beta + 2A][(1-\alpha)\cos^2\lambda + k - 1]\}} \right], \tag{5.22}$$

$k = 2, 3 \ldots, n-1$. *Then*

$$|a_n| \le \frac{1}{(n-1)!} \prod_{k=2}^{n} \left| [(B-A)\beta + A](k-2) + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda \right|, \quad n > N+2. \tag{5.23}$$

for $n = 2, 3 \ldots, N+2$; *and*

$$|a_n| \le \frac{1}{(N+1)!(n-1)} \prod_{k=2}^{N+3} \left| [(B-A)\beta + A](k-2) + (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda \right|, \quad n > N+2. \tag{5.24}$$

(b) *If, on the other hand,*

$$\beta(1-\alpha)(k-\alpha)\cos^2\lambda \le \frac{(k-1)}{(B-A)^2\beta}\{(k-1)(1-A^2) - (B-A)\beta[(B-A)\beta + 2A] \cdot [(1-\alpha)\cos^2\lambda + k - 1]\}, \tag{5.25}$$

then

$$|a_n| \le \frac{(B-A)\beta(1-\alpha)\cos\lambda}{(n-1)} \quad \textit{for} \quad n \ge 2. \tag{5.26}$$

The bounds in (5.23) *and* (5.26) *are sharp for all admissible* α, β, λ, A, B, *and for each* n.

Proof. Since $f(z) \in S^\lambda(\alpha, \beta, A, B)$, (5.7) gives

$$[[(B-A)\beta + A]zf'(z) - \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\} f(z)]\, w(z) = f(z) - zf'(z), \qquad w(z) \in \Omega. \tag{5.27}$$

Rewriting (5.27) in the form:

$$\left[[(B-A)\beta + A][z + \sum_{k=2}^{\infty} k a_k z^k] - \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\} \cdot [z + \sum_{k=2}^{\infty} a_k z^k] \right] w(z) = \sum_{k=2}^{\infty} (1-k) a_k z^k,$$

or, equivalently,

$$\Big[(B-A)\beta(1-\alpha)ze^{-i\lambda}\cos\lambda+\sum_{k=2}^{\infty}\{[(B-A)\beta+A](k-1)$$
$$+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}\,a_kz^k\Big]\,w(z)$$
$$=\sum_{k=2}^{\infty}(1-k)a_kz^k. \tag{5.28}$$

where $w(z)$ is given, as before, by (5.1).

Equating corresponding coefficients on both sides of (5.28) we observe that the coefficients a_n on the right-hand side depends only on the coefficients $a_2, a_3, \ldots, a_{n-1}$ $(n \geq 2)$ occuring on the left-hand side. Hence for $n \geq 2$, it follows from (5.28) that

$$\Big[(B-A)\beta(1-\alpha)ze^{-i\lambda}\cos\lambda+\sum_{k=2}^{n-1}\{[(B-A)\beta+A](k-1)$$
$$+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}\,a_kz^k\Big]\,w(z)$$
$$=\sum_{k=2}^{n}(1-k)a_kz^k+\sum_{k=n+1}^{\infty}b_kz^k, \tag{5.29}$$

where $\sum_{k=n+1}^{\infty} b_kz^k$ converges in U. Writing $z=re^{i\theta}$, integrating from 0 to 2π and using the bound $|w(z)| \leq |z|$ for $z \in U$, (5.29) gives

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|(B-A)\beta(1-\alpha)re^{i\theta}e^{-i\lambda}\cos\lambda+\sum_{k=2}^{n-1}\{[(B-A)\beta+A](k-1)$$
$$+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}\,a_kr^ke^{ik\theta}\Big|^2\,d\theta$$
$$\geq\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=2}^{n}(1-k)a_kr^ke^{ik\theta}+\sum_{k=n+1}^{n}b_kr^ke^{ik\theta}\right|^2\,d\theta,$$

which by Parseval's identity [24] is equivalent to

$$(B-A)^2\beta^2(1-\alpha)^2r^2\cos^2\lambda+\sum_{k=2}^{n-1}|[(B-A)\beta+A](k-1)$$
$$+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\Big|^2\,|a_k|^2$$
$$\geq\sum_{k=2}^{n}(1-k)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}. \tag{5.30}$$

Since the infinite series in (5.30) is non-negative for $0<r<1$, we have as $r \to 1$

$$(n-1)^2|a_n|^2\leq(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda+\sum_{k=2}^{n-1}\{|[(B-A)\beta+A][(k-1)$$
$$+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\Big|^2-(k-1)^2\Big\}\,|a_k|^2,\quad n\geq 2. \tag{5.31}$$

The following two cases will now arise:

(a) Let the inequality (5.21) hold. Suppose also that $n \le N+2$ in (5.31), where N is given by (5.22). Then, for $n = 2$, (5.31) immediately yields

$$|a_2| \le (B-A)\beta(1-\alpha)\cos\lambda, \tag{5.32}$$

which proves (5.23) for $n = 2$. We establish (5.23) for $n \le N+2$, from (5.31), by mathematical induction.

Fix n, $n \ge 3$, and suppose that (5.23) holds for $k = 2, 3 \ldots, n-1$. Then it follows from (5.31) that

$$|a_n|^2 \le \frac{1}{(n-1)^2}\left\{(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{k=2}^{n-1}[|[(B-A)\beta+A](k-1) + M|^2 - (k-1)^2]\,\frac{1}{((k-1)!)^2}\prod_{j=2}^{k}|[(B-A)\beta+A](j-2)+M|^2\right\}, \tag{5.33}$$

where $M = (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda$.
We must show that the square of the right-hand side of (5.23) is equal to the right-hand side of (5.33); that is,

$$\frac{1}{((m-1)!)^2}\prod_{j=2}^{m}|[(B-A)\beta+A](j-2)+M|^2$$
$$= \frac{1}{(m-1)^2}\left\{(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda + \sum_{k=2}^{m-1}\left[|[(B-A)\beta+A](k-1)+M|^2 - (k-1)^2\right]\cdot\frac{1}{((k-1)!)^2}\prod_{j=2}^{k}|[(B-A)\beta+A](j-2)+M|^2\right\}, \tag{5.34}$$

for $m = 3, 4, \ldots$. A brief calculation verifies (5.34) for $m = 3$ and proves (5.23) for $n = 3$. Assume that (3.23) is valid for all m, $3 < m \le n-1$; then (5.31) and (5.33) give

$$\begin{aligned}|a_n|^2 \le &\frac{1}{(n-1)^2}\Big\{(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda \\ &+\sum_{k=2}^{n-2}\left[\left|[(B-A)\beta+A](k-1)+M\right|^2-(k-1)^2\right]\cdot \\ &\cdot\frac{1}{((k-1)!)^2}\prod_{j=2}^{k}\left|[(B-A)\beta+A](j-2)+M\right|^2 \\ &+\left[\left|[(B-A)\beta+A](n-2)+M\right|^2-(n-2)^2\right]\cdot \\ &\cdot\frac{1}{((n-2)!)^2}\prod_{j=2}^{n-1}\left|[(B-A)\beta+A](j-2)+M\right|^2\Big\}\end{aligned}$$

$$
\begin{aligned}
=&\frac{(n-2)^2}{(n-1)^2}\left[\frac{1}{(n-2)^2}\left\{(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda\right.\right.\\
&+\sum_{k=2}^{n-1}\left[|[(B-A)\beta+A](k-1)+M|^2-(k-1)^2\right]\cdot\\
&\left.\cdot\frac{1}{((k-1)!)^2}\prod_{j=2}^{k}|[(B-A)\beta+A](j-2)+M|^2\right\}\\
&+\frac{1}{(n-2)^2}\left\{|[(B-A)\beta+A](n-2)+M|^2-(n-2)^2\right\}\cdot\\
&\left.\cdot\frac{1}{((n-2)!)^2}\prod_{j=2}^{n-1}|[(B-A)\beta+A](j-2)+M|^2\right]
\end{aligned}
$$

or

$$
\begin{aligned}
|a_n|^2\leq&\frac{(n-2)^2}{(n-1)^2}\left[\frac{1}{((n-2)!)^2}\prod_{j=2}^{n-1}|[(B-A)\beta+A](j-2)+M|^2\right]\\
&+\frac{1}{((n-1)!)^2}\cdot\left[|[(B-A)\beta+A](n-2)+M|^2-(n-2)^2\right]\cdot\\
&\cdot\left[\prod_{j=2}^{n-1}|[(B-A)\beta+A](j-2)+M|^2\right]\\
=&\frac{1}{((n-1)!)^2}\left[|[(B-A)\beta+A](n-2)+M|^2\right]\cdot\\
&\cdot\left[\prod_{j=2}^{n-1}|[(B-A)\beta+A](j-2)+M|^2\right]\\
=&\frac{1}{((n-1)!)^2}\prod_{j=2}^{n}|[(B-A)\beta+A](j-2)+M|^2.
\end{aligned}
$$

Thus, we get

$$
|a_n|\leq\frac{1}{(n-1)!}\prod_{j=2}^{n}|[(B-A)\beta+A](j-2)+M|,
$$

which completes the proof of (5.23).

With a view to prove the assertion (5.24) of Theorem 6, suppose that $n>N+2$, where N is given by (5.22). Then, retaining only the terms of the series on the right-hand side of (5.31) from $k=2$ to $k=N+2$, we have

$$
\begin{aligned}
(n-2)^2|a_n|^2\leq&(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda\\
&+\sum_{k=2}^{N+2}\left\{|[(B-A)\beta+A](k-1)+M|^2-(k-1)^2\right\}|a_k|^2,
\end{aligned}
\tag{5.35}
$$

where we have obviously dropped the non-negative termes from $k = N+3$ to $k = n-1$.

Now we substitute in (5.35) the upper bounds for

$$|a_2|,\ |a_3|,\ \ldots,\ |a_{N+2}|,$$

given by (5.23), and the assertion (5.24) follows upon simplifying the resulting equation.

(b) Let the inequality (5.25) hold. Then it follows from (5.31) that

$$(n-1)^2|a_n|^2 \leq (B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda \quad \text{for} \quad n \geq 2,$$

which evidently proves the assertion (5.26) of Theorem 6.

The bound in (5.23) is sharp for the function $f(z)$ given by

$$f(z) = z\{1-[(B-A)\beta+A]z\}^{-\frac{(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A]}},\ \beta \neq (\frac{A}{A-B}), \tag{5.36}$$

and the bounds in (5.26) are sharp for the functions $f_n(z)$ given by

$$f_n(z) = \begin{cases} z\{1-[(B-A)\beta+A]z^{n-1}\}^{\frac{-[(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A](n-1)}} & (\beta \neq (\frac{A}{A-B}) : n \geq 2) \\ z\exp\left[\frac{-A(1-\alpha)z^{n-1}e^{-i\lambda}\cos\lambda}{(n-1)}\right] & (\beta = (\frac{A}{A-B}); n \geq 2). \end{cases} \tag{5.37}$$

An immediate consequence of Theorem 6 may be stated as

Corollary 5. *Let the function $f(z)$ defined by (1.1) be in the class $C^{\lambda}(\alpha,\beta,A,B)$. Then, under the hypotheses (5.21) and (5.22).*

$$|a_n| \leq \frac{1}{n!}\prod_{k=2}^{n}\left|[(B-A)\beta+A][(k-2)+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|, \tag{5.38}$$

for $n = 2, 3, \ldots, N+2$; and

$$|a_n| \leq \frac{1}{(N+1)!n(n-1)}\prod_{k=2}^{N+3}\left|[(B-A)\beta+A][(k-2)\right.$$
$$\left.+(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\right|, \quad n > N+2. \tag{5.39}$$

(b) *If, on the other hand, the condition (5.25) holds true, then*

$$|a_n| \leq \frac{(B-A)\beta(1-\alpha)\cos\lambda}{n(n-1)} \quad \textit{for} \quad n \geq 2. \tag{5.40}$$

The estimates in (5.38) are sharp for the function $f(z)$ given by

$$1+\frac{zf''(z)}{f'(z)} = \frac{1-\{[(B-A)\beta+A]-(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}z}{1-[(B-A)\beta+A]z}$$
$$(\beta \neq (\frac{A}{A-B})), \tag{5.41}$$

while the estimates in (5.40) *are sharp for the functions* $f_n(z)$ *given by*

$$f_n'(z) = \begin{cases} \{1 - [(B-A)\beta + A]z^{n-1}\}^{\frac{-(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A](n-1)}} & (\beta \neq (\frac{A}{A-B}) : n \geq 2) \\ \exp\left[\frac{-A(1-\alpha)z^{n-1}e^{-i\lambda}\cos\lambda}{(n-1)}\right] & (\beta = (\frac{A}{A-B}); n \geq 2). \end{cases} \quad (5.42)$$

6. Radius of γ-Spiral and γ-Convex

Let S_1 be the family of all normalized functions which are analytic and univalent in U. In [18], Libera introduced the concept of "γ-spiral radius" for the classes of univalent functions as follows:

Definition 3. If $f(z) \in S_1$ and $|\gamma| < \frac{\pi}{2}$, then the γ-spiral radius of $f(z)$ is

$$\gamma - \text{s.r.}\{f(z)\} = \sup\left\{r : \text{Re}(e^{i\gamma}\frac{zf'(z)}{f(z)}) > 0, |z| < r\right\}. \quad (6.1)$$

Definition 4. If $F \subset S_1$ and $|\gamma| < \frac{\pi}{2}$, then the γ-spiral radius of F is

$$\gamma - \text{s.r.}F = \inf_{f \in F}[\gamma - \text{s.r.}\{f(z)\}]. \quad (6.2)$$

Also in [22], Mogra introduced the concept of "γ-convex radius" as follows:

Definition 5. If $f(z) \in S$, the class of analytic functions in U, and $|\gamma| < \frac{\pi}{2}$, then the γ-convex radius of $f(z)$ is

$$\gamma - \text{c.r.}\{f(z)\} = \sup\left\{r : \text{Re}\left\{e^{i\gamma}(1 + \frac{zf''(z)}{f'(z)})\right\} > 0, |z| < r\right\}. \quad (6.3)$$

Definition 6. If $G \subset S$ and $|\gamma| < \frac{\pi}{2}$, then the γ-convex radius of G is

$$\gamma - \text{c.r.}G = \inf_{f \in G}[\gamma - \text{c.r.}\{f(z)\}]. \quad (6.4)$$

Theorem 7. γ-s.r.$S^\lambda(\alpha, \beta, A, B)$, *is the smallest positive root* r_0 *of the equation*

$$[(B-A)\beta + A]\{(B-A)\beta(1-\alpha)\cos(\gamma - \lambda)\cos\lambda - [(B-A)\beta + A]\cos\gamma\}r^2 - (B-A)\beta(1-\alpha)r\cos\lambda + \cos\gamma = 0. \quad (6.5)$$

The result is sharp.

Proof. Let $f(z) \in S^{\lambda}(\alpha, \beta, A, B)$. Then by using Lemma 1, we have

$$\frac{zf'(z)}{f(z)} = \frac{1 + \{[(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}w(z)}{1 + [(B-A)\beta + A]w(z)}, \tag{6.6}$$

where $w(z)$ satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$. If $B(z) = e^{i\gamma}\frac{zf'(z)}{f(z)}$ and $|\gamma| < \frac{\pi}{2}$, then (6.6) may be written as

$$w(z) = \frac{e^{i\gamma} - B(z)}{[(B-A)\beta + A]B(z) - e^{i\gamma}([(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda)}, z \in U. \tag{6.7}$$

Now by applying Schwarz's Lemma [15], it follows that $B(z)$ maps the disc $|z| \leq r$ onto a disc $|B(z) - \eta| < R$, where

$$\eta = \frac{e^{i\gamma}\{1 - [(B-A)\beta + A]([(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda)r^2\}}{1 - [(B-A)\beta + A]^2 r^2}, \tag{6.8}$$

and

$$R = \frac{(B-A)\beta(1-\alpha)r\cos\lambda}{1 - [(B-A)\beta + A]^2 r^2}. \tag{6.9}$$

Hence $\mathrm{Re}(e^{i\gamma}\frac{zf'(z)}{f(z)}) \geq 0$ if and only if

$$\mathrm{Re}\left\{\frac{e^{i\gamma}\{1 - [(B-A)\beta + A]([(B-A)\beta + A] - (B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda)r^2\}}{1 - [(B-A)\beta + A]^2 r^2}\right\} \geq \frac{(B-A)\beta(1-\alpha)r\cos\lambda}{1 - [(B-A)\beta + A]^2 r^2}, \tag{6.10}$$

which, on simplification, and with the aid of (6.2) concludes the proof of Theorem 7.

The result is sharp for the function $f(z)$ given by

$$f(z) = \begin{cases} z\{1 - [(B-A)\beta + A]z\}^{\frac{-[(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A]}}, & \beta \neq (\frac{A}{A-B}), \\ z\exp\left[-A(1-\alpha)z\,e^{-i\lambda}\cos\lambda\right], & \beta = (\frac{A}{A-B}), \end{cases} \tag{6.11}$$

and

$$\xi = \frac{r\{[(B-A)\beta + A]r - e^{i(\lambda-\gamma)}\}}{1 - [(B-A)\beta + A]r\,e^{i(\lambda-\gamma)}}. \tag{6.12}$$

Replacing $(\frac{zf'(z)}{f(z)})$ by $(1 + \frac{zf''(z)}{f'(z)})$ in Theorem 7 and using Definition 6, we get the following result for the class $C^{\lambda}(\alpha, \beta, A, B)$.

Corollary 6. *γ-c.r. $C^\lambda(\alpha,\beta,A,B)$ is the smallest positive root r_0 of the equation* (6.5). *The result is sharp for the function $f(z)$ given by*

$$f'(z) = \begin{cases} \{1-[(B-A)\beta+A]z\}^{\frac{-(B-A)\beta(1-\alpha)e^{-i\lambda}\cos\lambda}{[(B-A)\beta+A]}}, & \beta \neq (\frac{A}{A-B}), \\ \exp\left[-A(1-\alpha)z\,e^{-i\lambda}\cos\lambda\right], & \beta = (\frac{A}{A-B}), \end{cases} \tag{6.13}$$

ξ being defined (as before) by (6.12).

7. Radius of Starlikeness and Convexity

We first state and prove

Theorem 8. *The sharp radius of starlikes of the class $S^\lambda(\alpha,\beta,A,B)$, $\beta \neq (\frac{A}{A-B})$, is given by*

$$r_s = 2\left\{(B-A)\beta(1-\alpha)\cos\lambda + \left[(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda - 4[(B-A)\beta+A]^2 \cdot \left[\frac{(B-A)\beta(1-\alpha)\cos^2\lambda}{[(B-A)\beta+A]} - 1\right]\right]^{\frac{1}{2}}\right\}^{-1}. \tag{7.1}$$

The expression in (7.1) *is real and finite only when $\beta \neq (\frac{A}{A-B})$ and such that*

$$(B-A)^2\beta^2(1-\alpha)^2\cos^2\lambda \geq 4[(B-A)\beta+A]^2\left[\frac{(B-A)\beta(1-\alpha)\cos^2\lambda}{[(B-A)\beta+A]} - 1\right]. \tag{7.2}$$

Proof. From (4.5), we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1-(B-A)\beta(1-\alpha)r\cos\lambda + [(B-A)\beta+A]^2\left[\frac{(B-A)\beta(1-\alpha)\cos^2\lambda}{[(B-A)\beta+A]} - 1\right]r^2}{1-[(B-A)\beta+A]^2r^2},$$

where $|z| = r$.
Thus $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ for $|z| < r_s$, where r_s is given by (7.1), provided that $\beta \neq (\frac{A}{A-B})$ and the condition (7.2) is satisfied.

To show that (7.1) is sharp, we let $f(z)$ be given by (5.36) and put

$$\xi = \frac{r\{[(B-A)\beta+A]r - e^{i\lambda}\}}{1-[(B-A)\beta+A]r\,e^{i\lambda}}. \tag{7.3}$$

we thus obtain

$$\frac{\xi f'(\xi)}{f(\xi)} = \frac{1-(B-A)\beta(1-\alpha)r\cos\lambda-[(B-A)\beta+A]^2\left[\frac{(B-A)\beta(1-\alpha)\cos^2\lambda}{[(B-A)\beta+A]}-1\right]r^2}{1-[(B-A)\beta+A]^2r^2},$$

which obviously has a zero real part when r is given by (7.1). This completes the proof of Theorem 8.

Making use of the relationship (1.6) between the classes $S^\lambda(\alpha,\beta,A,B)$ and $C^\lambda(\alpha,\beta,A,B)$, we can deduce the following consequence of Theorem 8.

Corollary 7. *The sharp radius of convexity of the class $C^\lambda(\alpha,\beta,A,B)$, $\beta \neq (\frac{A}{A-B})$, is given by (7.1). The result is sharp for the function $f(z)$ given by (5.41), ξ being defind (as before) by (7.3).*

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