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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE LIPSCHITZ CLASS BY (e,c) MEANS

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Abstract. In the present paper, we obtain the degree of approximation of $f\varepsilon\text{Lip}\alpha$ $(0<\alpha\le 1)$ by (e,c) means $(c>0)$ of its Fourier Series.

1. Introduction

Let $C_{2\pi}$ be the space of all 2π-periodic and continuous functions defined on $[-\pi,+\pi]$, which is a Banach space under the "sup" norm.

A function $f\varepsilon\text{Lip}\alpha$ $(0<\alpha\le 1)$ if

$$f(x+h)-f(x)=O(|h|^\alpha) \tag{1}$$

For each $f\varepsilon c_{2\pi}$ let the Fourier series be given by

$$S(x)=\sum_{m=-\infty}^{\infty} C_m \exp(i\,m\,x) \tag{2}$$

where, $C_{m's}$ are Fourier coefficients. Let the n^{th} partial sum of the series (2) be

$$S_n(f;x)=\sum_{m=-n}^{n} C_m \exp(i\,m\,x)$$

A series $\sum_{n=0}^{\infty} a_n$ with the sequence of partial sum $\{S_n\}$ is said to be summable (e,c), $(c>0)$ if (see [4])

$$\lim_{n\to\infty} t_n^c=\lim_{n\to\infty}\sqrt{\frac{c}{\pi n}}\sum_{k=-\infty}^{\infty}\exp(-\frac{ck^2}{n})S_{n+k} \tag{3}$$

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exists, where it is to be understood that $S_{n+k} = 0$, when $n + k < 0$.

The (e, c) summability method is regular for $c > 0$. It is interesting to note that this method includes several other methods of summability, namely, Borel, (E, q), (γ, k) etc. (For details see Hardy [3] Theorem 159).

We shall write

$$||t_n^c - f|| = \sup_{-\pi \leq x \leq \pi} |t_n^c(f; x) - f(x)| \tag{4}$$

where $t_n^c(f; x)$ is $n^{th}(e, c)$-means of the Fourier series of f at x.

Degree of approximation by Borel means and (E, q)-means were obtained by Chandral [1]&[2] respectively. Since (e, c)-method includes (B, α) and (E, q) method, it is natural to ask, what will be the result if we apply (e, c)-means to obtain the degree of approximation for $f \varepsilon \mathrm{Lip}\alpha$ $(0 < \alpha \leq 1)$?

We shall prove the following theorem.

Theorem: *Let* $f \varepsilon c_{2\pi} \cap \mathrm{Lip}\alpha$, $0 < \alpha \leq 1$
Then

$$||t_n^c - f|| = O(n^{-\alpha/2})$$

2. Inequalities

In the proof of our theorem, we shall use the following inequalities

$$\sum_{k=n+1}^{\infty} k \exp(-\frac{ck^2}{n}) \leq \frac{n}{2c} \exp(-cn) \tag{5}$$

$$\left| \sum_{k=n+1}^{\infty} \exp(-\frac{ck^2}{n}) \sin(n + k + \frac{1}{2})t \right| \leq \frac{nt}{2c} \exp(-cn) \tag{6}$$

$$\sum_{k=n+1}^{\infty} \exp(-\frac{ck^2}{n}) \cos(kt) = O\left\{\frac{\exp(-cn)}{t}\right\} \tag{7}$$

$$1 + 2\sum_{k=1}^{\infty} \exp(-\frac{ck^2}{n}) \cos(kt) = \sqrt{\frac{\pi n}{c}} \{\exp(-\frac{nt^2}{4c}) + O(\exp(-\frac{n\pi}{4c}))\} \tag{8}$$

The inequality (6) follows from (5), (7) may be obtained by using Abel's Lemma and (8) may be obtained by the classical formula for theta function (see Siddiqui [5]). Thus we prove only (5).

Proof of (5) We observe that $Y \exp(-\frac{cY^2}{n})$ is non-increasing for $Y \geq \sqrt{\frac{n}{2c}}$ and hence

$$\sum_{k=n+1}^{\infty} k \exp(-\frac{ck^2}{n}) \leq \int_n^{\infty} Y \exp(\frac{c\gamma^2}{n})d\gamma$$
$$=\frac{n}{2c}\int_n^{\infty} \frac{d}{d\gamma}(-\exp(-\frac{c\gamma^2}{n}))d\gamma$$
$$=\frac{n}{2c}\exp(-cn)$$

This completes the proof of (5)

3. Proof of Theorem

Following Titchmarsh [6] p.403, we have

$$S_n(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi_x(t)}{\sin(\frac{t}{2})}\sin(n+\frac{1}{2})t\,dt$$

where

$$2\varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$$

then

$$t_n^c(f;x) - f(x) = \frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi} \frac{\varphi_x(t)}{\sin(\frac{t}{2})}\left[\sum_{k=-n}^{\infty} \exp(-\frac{ck^2}{n})\sin(n+k+\frac{1}{2})t\right] dt$$
$$=\frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\int_0^{\pi} \frac{\varphi_x(t)}{\sin(\frac{t}{2})}\left[\left\{1+2\sum_{k=1}^{n}\exp(-\frac{ck^2}{n})\cos(kt)\right\}\sin(n+\frac{1}{2})t\right.$$
$$\left.+\sum_{k=n+1}^{\infty}\exp(-\frac{ck^2}{n})\sin(n+k+\frac{1}{2})t\right] dt$$
$$=\frac{1}{\pi}\sqrt{\frac{c}{\pi n}}\left[\int_0^{\pi} \frac{\varphi_x(t)}{\sin(\frac{t}{2})}\left[\left\{1+2\sum_{k=1}^{n}\exp(-\frac{ck^2}{n})\cos(kt)\right\}\sin(n+\frac{1}{2})t\right.\right.$$
$$\left.\left.-2\sum_{k=n+1}^{\infty}\exp(-\frac{ck^2}{n})\cos(kt)\sin(n+\frac{1}{2})t+\sum_{k=n+1}^{\infty}\exp(-\frac{ck^2}{n})\sin(n+k+\frac{1}{2})t\right] dt\right]$$
$$=I_1 + I_2 + I_3,\ \text{say}$$

Hence

$$||t_n^c(f) - f|| \leq ||I_1|| + ||I_2|| + ||I_3|| \tag{9}$$

Now

$$||I_3|| = O(n^{-1/2})\int_0^{\pi} t^{\alpha-1}\left|\sum_{k=n+1}^{\infty}\exp(-\frac{ck^2}{n})\sin(n+k+\frac{1}{2})t\right| dt$$

$$
\begin{aligned}
&=O(n^{1/2})\int_0^{\frac{\pi}{n}} t^{\alpha}\exp(-cn)dt + O(n^{-1/2})\int_{\frac{\pi}{n}}^{\pi} t^{\alpha-2}\exp(-cn)\,dt \\
&=O(n^{-\frac{1}{2}-\alpha}\exp(-cn)) + O(n^{-\frac{1}{2}})\exp(-cn)\begin{cases} n^{1-\alpha} & \text{for } 0<\alpha<1 \\ \log n & \text{for } \alpha=1 \end{cases} \\
&=\begin{cases} O(n^{\frac{1}{2}-\alpha}\exp(-cn)) & \text{for } 0<\alpha<1 \\ O(n^{-\frac{1}{2}}\exp(-cn)\log n) & \text{for } \alpha=1 \end{cases}
\end{aligned} \tag{10}
$$

Similarly

$$
||I_2|| = \begin{cases} O(n^{\frac{1}{2}-\alpha}\exp(-cn)) & \text{for } 0<\alpha<1 \\ \\ O(n^{-\frac{1}{2}}\exp(-cn)\log n) & \text{for } \alpha=1 \end{cases} \tag{11}
$$

Finally by (8), we have

$$
\begin{aligned}
I_1 &= \int_0^{\pi} \frac{\varphi_x(t)}{\sin(t/2)}\sin(n+\frac{1}{2})t\,\exp(-\frac{nt^2}{4c})dt + O(1)\int_0^{\pi}\frac{|\varphi_x(t)}{\sin(t/2)}\left|\sin(n+\frac{1}{2})t\right|\exp(-\frac{n\pi}{4c})dt \\
&=I_{1,1}+I_{1,2}\ \text{say}
\end{aligned} \tag{12}
$$

Now

$$
\begin{aligned}
||I_{1,2}|| &=O(\exp(-\frac{n\pi}{4c}))\left[\int_0^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi}\right] t^{\alpha-1}\left|\sin(n+\frac{1}{2})t\right|\,dt \\
&=O\left\{n^{-\alpha}\exp(-\frac{n\pi}{4c})\right\} + O\{\exp(-\frac{n\pi}{4c})\}
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
||I_{1,1}|| &=O(1)\int_0^{\pi} t^{\alpha-1}\exp(-\frac{nt^2}{4c})\left|\sin(n+\frac{1}{2})t\right|\,dt \\
&=O(1)\int_0^{\frac{\pi}{\sqrt{n}}} t^{\alpha-1}dt + O(n^{-1})\int_{\frac{\pi}{\sqrt{n}}}^{\pi} t^{\alpha-2}\frac{\partial}{\partial t}\left[\exp(-\frac{nt^2}{4c})\right]dt \\
&=O(n^{-\alpha/2})
\end{aligned} \tag{14}
$$

Collection of (9), (10), ... (14) completes the proof of the theorem.

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