TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 3, Autumn 1995

ALMOST RIEMANN INTEGRABLE FUNCTIONS

MOUSTAFA DAMLAKHI

Abstract. An arbitrary function f on a bounded interval $[a, b]$ is termed an almost R-integrable function if there exists a Riemann integrable function g such that $f = g$ a.e. In this note a characterization of the class of almost R-integrable functions is obtained.

1. Introduction.

A bounded function $f(x)$ defined on a bounded interval $[a, b]$ is Riemann integrable if and only if $f(x)$ is continuous a.e. But a function which is a.e. equal to a Riemann integrable function on $[a, b]$ need not be Riemann integrable as the characteristic function of the set of rational points in the interval shows. In this note, we characterize the class of functions on $[a, b]$ which are a.e. equal to Riemann integrable functions.

Here, the measure and the integral (unless otherwise specified) are taken in the sense of Lebesgue.

2. Almost R-integrable functions:

Definition 1. An arbitrary function defined on a bounded interval $[a, b]$ is said to be *almost R-integrable* if there exists a bounded function $g(x)$ on $[a, b]$ integrable in the sense of Riemann such that $f(x) = g(x)$ a.e.

Example of an unbounded function in $[0, 1]$ that is almost R-integrable is

$$f(x) = \begin{cases} n & \text{if } x = \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Note, however,

$$g(x) : \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \epsilon (0, 1) \\ 0 & \text{if } x = 0 \end{cases}$$

Received January 18, 1994 revised April 8, 1994.
1991 *Mathematics Subject Classification* 26A42
Key words and phrases. Almost integrability.

is Riemann integrable in the extended sense but not almost R-integrable

Proposition 2. *An almost R-integrable function on $[a, b]$ is integrable (in the sense of Lebesgue) but not conversely.*

Proof. It is obvious that f is integrable if it is almost R-integrable.

Now, we exhibit an integrable function that is not almost R-integrable.

Let A be a nowhere dense perfect set of measure $\frac{1}{2}$ in $[0, 1]$.

If $f(x)$ is the characteristic function of A, then f is integrable, but not almost R-integrable.

For if $f = g$ a.e. where g is Riemann integrable, we arrive at a contradiction since g is continuous a.e. while f is not continuous at every one of the points of A which is of measure $\frac{1}{2}$.

3. The class $\mathcal{E}$ of functions:

Let $\mathcal{E}$ denote the class of functions f on $[a, b]$ such that f is the limit a.e. of an increasing sequence (φ_n) of step functions.

Theorem: *A function f defined on $[a, b]$ is almost R-integrable if and only if $f \epsilon \mathcal{E} \cap -\mathcal{E}$.*

Proof: Since every Riemann integrable function is in $\mathcal{E}$, it is clear that every almost R-integrable function $f \epsilon \mathcal{E} \cap -\mathcal{E}$.

Conversely, let us suppose that $f \epsilon \mathcal{E} \cap -\mathcal{E}$. Then f^+ and $f^- \epsilon \mathcal{E} \cap -\mathcal{E}$; we'll therefore assume that f is positive.

There exists then, an increasing sequence that $\{\varphi_n\}$ of step functions and another decreasing sequence $\{\psi_n\}$ of step functions satisfying the conditions: sup $\varphi_n = f$ a.e. and inf $\psi_n = f$ a.e.

Remark that for some constants A and B and all n, we have $\int_a^b \varphi_n(x)dx \leq A$ and $\int_a^b \psi_n(x)dx \leq B$.

By Lebesgue's monotone convergence theorem we also have

$$\sup \int_a^b \varphi_n dx = \int_a^b f(x)dx = \inf \int_a^b \psi_n dx.$$

Consequently, given $\mathcal{E} > 0$, we can choose N so that

$$\int_a^b \psi_m dx - \int_a^b \varphi_n dx < \epsilon \quad \text{if} \quad n, m \geq N.$$

Now, by hypothesis $\varphi_n \leq \psi_1$ a.e. and since φ_n and ψ_1 are step functions, the sequence $\{\varphi_n\}$ is bounded above.

Let $g(x) = \sup \varphi_n(x)$ on $[a,b]$. then $g(x)$ is a bounded function such that $g = f$ a.e. Now, it is enough to show that g is Riemann integrable.

Let $\varphi_N(x) = \sum c_k X_{I_k}$

Take now a partition $P : (a = x_0 < x_1 < \cdots < x_n = b)$ of $[a,b]$ such that each I_k is one of the subintervals of P. Then if m_p is the lower Darboux sum of g corresponding to the partition P, we have

$$\int_a^b \varphi_N dx \leq m_p \leq \underline{\int_a^b} g(x)dx, \quad \text{where}$$

$\underline{\int_a^b} g(x)dx$ denotes the lower integral of g in the sense of Riemann.

In the same way, if $\inf \psi_n(x) = h(x)$ then $h(x) \geq g(x)$ because $\psi_m \geq \varphi_n$ for any n and m; also we can prove that $\int_a^b \psi_N dx \geq \overline{\int_a^b} h(x)dx \geq \overline{\int_a^b} g(x)dx$.

Consequently,

$$\overline{\int_a^b} g(x)dx - \underline{\int_a^b} g(x)dx \leq \int_a^b \psi_N dx - \int_a^b \varphi_N dx < \epsilon.$$

Since ϵ is arbitrary, we conclude that g is Riemann integrable

This completes the proof of the theorem.

Reference

[1] Soo Bong Chae, *Lebesgue integration*, Marcel Dekker, Inc. 1980.

[2] F. Riesz and B. Sz. Nagy, *Functional Analysis*, Ungar Publishing Co., Sixth printing, 1972.

Department of Mathematics, King Saud University P.O. Box 2455, Riyadh 11451, Saudi Arabia.