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HUA'S INEQUALITY FOR COMPLEX NUMBER

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Abstract. Variants for the complex numbers of the celebrated Lo-Keng Hua's inequality which is very important in Number Theory are given.

1. The following inequality due to Lo-Keng Hua is important in Number Theory [1]:

$$(\delta-\sum_{i=1}^{n}x_i)^2+\alpha\sum_{i=1}^{n}x_i^2\ge k_n\delta^2, \tag{1}$$

where $\delta,\alpha>0$, $x_i\in\mathbb{R}\,(i=\overline{1,n})$ and $k_n=\alpha(n+\alpha)^{-1}$ with equality if and only if $x_i=h_n\delta$ where $h_n=(n+\alpha)^{-1}$.

Recently, Chung-Lie Wang [2] gave the following interesting generalization of (1):

Theorem A. *Let α,δ be as above. Then for $p>1$, the inequality*

$$(\delta-\sum_{i=1}^{n}x_i)^p+\alpha^{p-1}\sum_{i=1}^{n}x_i^p\ge k_n^{p-1}\delta^p \tag{2}$$

holds for all nonnegative $x_i\in\mathbb{R}\,(i=\overline{1,n})$ with $\sum_{i=1}^{n}x_i\ge\delta$. The sign of inequality in (2) is reversed for $0<p<1$. In either case, the sign of equality holds in (2) iff $x_i=h_n\delta\,(i=\overline{1,n})$.

An intergral variant is also given.

2. In this paper, we shall give two variants of inequality (1) for complex numbers. The following lemma holds

Lemma. *Let $\alpha>0$ and $\delta,z\in\mathbb{C}$. Then the following inequality*

$$|\delta-z|^2+\alpha|z|^2\ge\frac{d|\delta|^2}{1+\alpha} \tag{3}$$

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holds. The equality is valid in (3) *iff*

$$z = \frac{1}{1+\alpha}\delta.$$

Proof. If $\delta = 0$, the inequality in (3) is obvious.

Suppose that $\delta \neq 0$. Then by the inequality $|\delta - z| \geq | \, |\delta| - |z| \, |$ and by Hua's result for $n = 1$ we have

$$|\delta - z|^2 + \alpha|z|^2 \geq (|\delta| - |z|)^2 + \alpha|z|^2 \geq \frac{\alpha|\delta|^2}{1+\alpha}. \tag{4}$$

The second equality in (4) holds (see Hua's result for $n = 1$) iff

$$|z| = \frac{1}{1+\alpha}|\delta|. \tag{5}$$

On the other hand, the first equality is valid in (4) iff

$$|\delta - z|^2 = | \, |\delta| - |z| \, |^2, \quad \text{i.e.,} \quad \mathrm{Re}(\overline{\delta} \cdot z) = |\delta| \, |z|. \tag{6}$$

Suppose that $z = \lambda\delta$ with $\lambda \in \mathbb{C}$. Then by (6) we have

$$|\delta|^2 \mathrm{Re}(\lambda) = |\lambda| \, |\delta|^2$$

and, since $\delta \neq 0$, we deduce

$$\mathrm{Re}(\lambda) = | \, \lambda \, | = ([\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2)^{1/2}$$

i.e., $\lambda \in \mathbb{R}$ and moreover, $\lambda > 0$, hence by the equality (5) we deduce

$$\lambda \mid \delta \mid = \frac{1}{\alpha+1} \mid \delta \mid$$

which give us $\lambda = \frac{1}{\alpha+1}$.

Consequently the equality holds in (3) iff $z = \frac{1}{\alpha+1}\delta$.

We can give the following variant of Hua's inequality for complex numbers.

Theorem 1. *Let* $\alpha > 0$ *and* $\delta, z_1, \ldots, z_n \in \mathbb{C}$. *Then the following inequality*

$$|\delta - \sum_{i=1}^{n} z_i|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha|\delta|^2}{n+\alpha} \tag{7}$$

holds. The equality is valid in (7) *iff:*

$$z_i = \frac{\delta}{n+\alpha} \qquad (i = \overline{1,n}).$$

Proof. By the well known Cauchy-Buniakowski-Schwarz's inequality for complex numbers:

$$n\sum_{i=1}^{n} | z_i |^2 \geq | \sum_{i=1}^{n} z_i |^2$$

with equality iff $z_i = \lambda \in \mathbb{C}$ for all $i \in \{1, \ldots, n\}$, one has

$$| \delta - \sum_{i=1}^{n} z_i |^2 + \alpha \sum_{i=1}^{n} | z_i |^2 \geq | \delta - \sum_{i=1}^{n} z_i |^2 + \frac{\alpha}{n} | \sum_{i=1}^{n} z_i |^2 .$$

Now, using the above lemma, we have:

$$| \delta - \sum_{i=1}^{n} z_i |^2 + \frac{\alpha}{n} | \sum_{i=1}^{n} z_i |^2 \geq \frac{\alpha | \delta |^2}{n+\alpha}$$

with equality iff

$$\sum_{i=1}^{n} z_i = \frac{n}{n+\alpha}\delta.$$

Consequently, the equality holds in (7) iff

$$n\lambda = \sum_{i=1}^{n} z_i = \frac{n}{n+\alpha}\delta$$

i.e., $\lambda = \frac{\delta}{n+\alpha} = z_i$ for all $i \in \{i, \ldots, n\}$, and the proof is finished.

3. The following generalization of inequality (7) also holds.

Theorem 2. *Let $\alpha > 0$ and $\delta, z_i, w_i \in \mathbb{C}\,(i = \overline{1,n})$. Then the following inequality:*

$$|\delta - \sum_{i=1}^{n} z_i w_i|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha|\delta|^2}{\alpha + \sum_{i=1}^{n} |w_i|^2} \tag{8}$$

holds. The equality is true in (8) iff

$$z_i = \frac{\delta \overline{w}_i}{\alpha + \sum_{i=1}^{n} |w_i|^2} \quad \text{for all} \quad i \in \{1, \ldots, n\}.$$

Proof. If $\sum_{i=1}^{n} |w_i|^2 = 0$, i.e., $w_i = 0$ for all $i \in \{1, \ldots, n\}$, the inequality (8) is valid.

Suppose that $\sum_{i=1}^{n} |w_i|^2 > 0$, then by Cauchy-Buniakowski-Schwarz's inequality

$$|\sum_{i=1}^{n} z_i w_i|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2$$

with equality iff $z_i = \lambda \overline{w}_i$ ($\lambda \in \mathbb{C}$) for all $i \in \{1, \ldots, n\}$, we have

$$|\delta - \sum_{i=1}^{n} z_i w_i|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq |\delta - \sum_{i=1}^{n} z_i w_i|^2 + \frac{\alpha}{\sum_{i=1}^{n} |w_i|^2} |\sum_{i=1}^{n} z_i w_i|^2.$$

Now, using the above lemma, we get:

$$|\delta - \sum_{i=1}^{n} z_i w_i|^2 + \frac{\alpha}{\sum_{i=1}^{n} |w_i|^2} |\sum_{i=1}^{n} z_i w_i|^2 \geq \frac{\alpha |\delta|^2}{\alpha + \sum_{i=1}^{n} |w_i|^2}$$

with equality iff

$$\sum_{i=1}^{n} z_i w_i = \frac{\sum_{i=1}^{n} |w_i|^2}{\alpha + \sum_{i=1}^{n} |w_i|^2} \delta.$$

But $z_i = \lambda \overline{w}_i \, (i = \overline{1, n})$ hence the equality holds in (8) iff

$$z_i = \lambda \overline{w}_i = \frac{\delta \overline{w}_i}{\alpha + \sum_{i=1}^{n} |w_i|^2} \qquad (i = \overline{1, n})$$

and the proof is finished.

Remark 1. If in (8) we put $w_i = 1 \, (i = \overline{1, n})$ we recapture (7).

Remark 2. Let $\alpha > 0$ and $\delta, z_i, w_i \in \mathbb{C}$ $(i = \overline{1, n})$ with $|w_i| = 1$ for all $i \in \{1, \ldots, n\}$. Then one has the inequality

$$|\delta - \sum_{i=1}^{n} z_i w_i|^2 + \alpha \sum_{i=1}^{n} |z_i|^2 \geq \frac{\alpha |\delta|^2}{\alpha + n}$$

with equality iff

$$z_i = \frac{\delta \overline{w}_i}{n + \alpha}, \qquad i \in \{1, \ldots, n\}$$

i.e., a class of Hua's type inequalities for complex numbers.

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