TAMKANG JOURNAL OF MATHEMATICS
Volume 26, Number 3, Autumn 1995

CR-SUBMANIFOLDS OF A QUASI-KAEHLER MANIFOLD

S. H. KON AND SIN-LENG TAN

Abstract. Let M be a CR-submanifold of a quasi-Kaehler manifold N. Sufficient conditions for the holomorphic distribution D in M to be integrable are derived. We also show that D is minimal. It follows that an (almost) complex submanifold of a quasi-Kaehler manifold is minimal, this generalizes the well known result that a complex submanifold of a Kaehler manifold is minimal.

1. Introduction

The notion of a CR-submanifold was introduced by A. Bejancu [1]. A CR-submanifold generalizes both a complex submanifold and a totally real submanifold. The geometry of a CR-submanifold in a Kaehler manifold has been extensively studied, a number of these results also hold for a CR-submanifold of a nearly Kaehler manifold, see [2], [6].

The integrability of the holomorphic distribution of a CR-submanifold in a quasi-Kaehler manifold will be considered in this paper. We will also show that the holomorphic distribution is minimal, consequently an (almost) complex submanifold of a quasi-Kaehler manifold is always minimal, this generalizes the well known result that a complex submanifold of a Kaehler manifold is minimal.

2. Preliminaries

Let N be an almost Hermitian manifold with Hermitian metric g and almost complex structure J. A real Riemannian manifold M, isometrically immersed in N, is said to be an (almost) complex submanifold if T_xM is invariant by J, i.e.

$$J(T_xM) = T_xM \quad \text{for each} \quad x \in M,$$

while M is said to be a totally real submanifold if T_xM is anti-invariant, i.e.

$$J(T_xM) \subset T_xM^{\perp} \quad \text{for each} \quad x \in M,$$

Received March 9, 1994; revised May, 17, 1994.
1991 *Mathematics Subject Classification.* 53B35, 53C15, 53C40.

where $T_xM^\perp$ is the normal space to M at x. More generally, M is said to be a CR-submanifold of N if there exists a differentiable distribution

$$D : x \longrightarrow D_x \subset T_xM$$

on M satisfying the following conditions:
(i) D is holomorphic, i.e. $J(D_x) = D_x$ for each $x \in M$,
(ii) the complementary orthogonal distribution

$$D^\perp : x \longrightarrow D_x^\perp \subset T_xM,$$

is anti-invariant, i.e. $J(D_x^\perp) \subset T_xM^\perp$ for each $x \in M$.
The distribution D will be referred to as the holomorphic distribution of the CR-submanifold M. An (almost) complex submanifold is clearly a CR-submanifold where $D_x = T_xM$.

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection on M and N respectively. Denote by $\Gamma(TN)$, the differentiable sections of TN, then we have

$$(\tilde{\nabla}_X J)Y = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y$$

and

$$(\tilde{\nabla}_X J)JY = -J(\tilde{\nabla}_X J)Y \quad \text{for} \quad X, Y \in \Gamma(TN).$$

Moreover, by the Gauss formula,

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{for} \quad X, Y \in \Gamma(TM),$$

where $h(X,Y) \in \Gamma(TM^\perp)$, h is a symmetric bilinear form on $\Gamma(TM)$ called the second fundamental form of M.

The fundamental 2-form Ω of an almost Hermitian manifold N is defined by

$$\Omega(X,Y) = g(X, JY) \quad \text{for} \quad X, Y \in \Gamma(TN).$$

It is easily verified that

$$\tilde{\nabla}_X \Omega(Y,Z) = g(Y, (\tilde{\nabla}_X J)Z)$$

and

$$3d\Omega(X,Y,Z) = g\left[(\tilde{\nabla}_X J)Z, Y\right] + g\left[(\tilde{\nabla}_Y J)X, Z\right] + g\left[(\tilde{\nabla}_Z J)Y, X\right]$$

for $X, Y, Z \in \Gamma(TN)$.
An almost Hermitian manifold N is said to be a
(i) Kaehler manifold if $(\tilde{\nabla}_X J)Y = 0$ for all $X, Y \in \Gamma(TN)$,
(ii) almost Kaehler manifold if $d\Omega(X,Y,Z) = 0$ for all $X, Y, Z \in \Gamma(TN)$,
(iii) nearly Kaehler manifold if $(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0$ for all $X, Y \in \Gamma(TN)$,
(iv) quasi-Kaehler manifold if $(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_{JX} J)JY = 0$ for all $X, Y \in \Gamma(TN)$.

It is easy to show that every almost Kaehler or nearly Kaehler manifold is necessarily a quasi-Kaehler manifold while a Kaehler manifold is both almost Kaehler and nearly Kaehler, see [5].

Finally, The Nijenhuis tensor field of J is defined by

$$[J,J](X,Y) = [JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY],$$

for $X, Y \in \Gamma(TN)$.

3. Integrability of the holomorphic distribution

From now on, M will be assumed to be a CR-submanifold of a quasi-Kaehler manifold N. We begin by collecting together a few simple results needed in our work later.

Proposition 3.1. *Let M be a CR-submanifold of a quasi-Kaehler manifold N, then we have*

(i) $[J,J](X,Y) = 2(\tilde{\nabla}_X J)JY - 2(\tilde{\nabla}_Y J)JX$
$= 2J(\tilde{\nabla}_Y J)X - 2J(\tilde{\nabla}_X J)Y$ for $X, Y \in \Gamma(TN)$,

(ii) $[JX,Y]+[X,JY] = \frac{1}{2}J[J,J](X,Y)+J[X,Y]+\nabla_{JX}Y-\nabla_{JY}X+h(JX,Y)-h(X,JY)$ for $X, Y \in \Gamma(D)$,

(iii) $h(X,JY) - h(JX,Y) = \frac{1}{2}J[J,J](X,Y) + J[X,Y] + \nabla_Y JX - \nabla_X JY$ for $X, Y \in \Gamma(D)$.

Proof. (i) Since the Levi-Civita connection is torsion free, we have for $X, Y \in \Gamma(TN)$,

$$\begin{aligned}
&[J,J](X,Y)\\
&=[JX,JY] - [X,Y] - J[JX,Y] - J[X,JY]\\
&=\tilde{\nabla}_{JX}JY - \tilde{\nabla}_{JY}JX - \tilde{\nabla}_X Y + \tilde{\nabla}_Y X - J\tilde{\nabla}_{JX}Y + J\tilde{\nabla}_Y JX - J\tilde{\nabla}_X JY + J\tilde{\nabla}_{JY}X\\
&=(\tilde{\nabla}_{JX}J)Y - (\tilde{\nabla}_{JY}J)X + J(\tilde{\nabla}_Y J)X - J(\tilde{\nabla}_X J)Y\\
&= - (\tilde{\nabla}_{JX}J)J^2Y + (\tilde{\nabla}_{JY})J^2X - (\tilde{\nabla}_Y J)JX + (\tilde{\nabla}_X J)JY\\
&=2(\tilde{\nabla}_X J)JY - 2(\tilde{\nabla}_Y J)JX \text{ since } N \text{ is quasi- Kaehler}\\
&=2J(\tilde{\nabla}_Y J)X - 2J(\tilde{\nabla}_X J)Y.
\end{aligned}$$

(ii) For $X, Y \in \Gamma(D)$,

$$\begin{aligned}
&\frac{1}{2}J[J,J](X,Y) + J[X,Y] + \nabla_{JX}Y - \nabla_{JY}X + h(JX,Y) - h(X,JY) \\
=&\frac{1}{2}J[J,J](X,Y) + J[X,Y] + \tilde{\nabla}_{JX}Y - \tilde{\nabla}_{JY}X \\
=&\frac{1}{2}J[J,J](X,Y) + J[X,Y] + \tilde{\nabla}_{Y}JX + [JX,Y] - \tilde{\nabla}_{X}JY - [JY,X] \\
=&\frac{1}{2}J[J,J](X,Y) + J[X,Y] + [JX,Y] + [X,JY] + (\tilde{\nabla}_{Y}J)X + J\tilde{\nabla}_{Y}X \\
&\qquad - (\tilde{\nabla}_{X}J)Y - J\tilde{\nabla}_{X}Y \\
=&\frac{1}{2}J[J,J](X,Y) + [JX,Y] + [X,JY] - J\left(J(\tilde{\nabla}_{Y}J)X - J(\tilde{\nabla}_{X}J)Y\right) \\
=&[JX,Y] + [X,JY] \text{ using (i)}
\end{aligned}$$

(iii) This follows from (ii) since ∇ is torsion free.

We can now generalize some results on the integrability of the holomorphic distribution of a CR-submanifold of a nearly Kaehler manifold to the setting of a quasi-Kaehler manifold. We first generalize a result of Sato (see [2, page 26]).

Theorem 3.2. *Let M be a CR-submanifold of a quasi-Kaehler manifold N. Then the holomorphic distribution D is integrable if and only if the following conditions are satisfied:*

$$h(X,JY) = h(JX,Y) \text{ and } [J,J](X,Y) \in \Gamma(D) \text{ for } X,Y \in \Gamma(D).$$

Proof. Suppose D is integrable, then $[J,J](X,Y) \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$, see [2, page 25] and so

$$h(X,JY) - h(JX,Y) = \frac{1}{2}J[J,J](X,Y) + J[X,Y] + \nabla_Y JX - \nabla_X JY \in \Gamma(TM)$$

showing that $h(X,JY) = h(JX,Y)$.
Conversely suppose that $h(X,JY) = h(JX,Y)$ and $[J,J](X,Y) \in \Gamma(D)$ for $X, Y \in \Gamma(D)$. Then we have $J[X,Y] = -\frac{1}{2}J[J,J](X,Y) + \nabla_Y JY - \nabla_Y JX \in \Gamma(TM)$.
Moreover for any $U \in \Gamma(D^{\perp})$ we have $Z \in \Gamma(TM^{\perp})$ such that $U = JZ$ and hence

$$g(U, J[X,Y] = -g(JU,[X,Y]) = g(Z,[X,Y]) = 0.$$

This shows that $J[X,Y] \in \Gamma(D)$ for $X, Y \in \Gamma(D)$ and consequently D is integrable.

Combining the above with Proposition 3.1 (i) we can reformulate the conditions for D to be integrable as follow.

Theorem 3.3. *Let M be a CR-submanifold of a quasi-Kaehler manifold N. The holomorphic distribution D is integrable if and only if*

$$h(X, JY) = h(JX, Y) \text{ and } (\tilde{\nabla}_X J)Y - (\tilde{\nabla}_Y J)X \in \Gamma(D) \text{ for } X, Y \in \Gamma(D).$$

4. Minimal Distributions

Let D be a differentiable distribution on a Riemannian manifold with Levi-Civita connection ∇. We write

$\mathring{h}(X, Y) = (\nabla_X Y)^\perp$ for $X, Y \in \Gamma(D)$ where $(\nabla_X Y)^\perp$ denotes the component of $\nabla_X Y$ in the orthogonal complementary distribution $D^\perp$. If $\{X_1, \ldots, X_r\}$ is an orthonormal basis for D we define the mean curvature vector $\mathring{H}$ of D by

$$\mathring{H} = \frac{1}{r} \sum_{k=1}^{r} \mathring{h}(X_k, X_k).$$

The distribution D is said to be minimal if the mean curvature vector $\mathring{H}$ vanishes identically.

For the holomorphic distribution D, we can find an orthonormal basis of the form $\{X_1, \ldots, X_s, JX_1, \ldots, JX_s\}$ where $r = 2s = \dim_{\mathbb{R}} D$ since D is invariant under J. Hence

$$\mathring{H} = \frac{1}{r} \sum_{k=1}^{s} (\mathring{h}(X_k, X_k) + \mathring{h}(JX_k, JX_k))$$

and so D is minimal in M if $\nabla_{X_k} X_k + \nabla_{JX_k} JX_k$ has no component in $D^\perp$ for $k = 1, \ldots, s$.

When M is a CR-submanifold of a Kaehler manifold, the holomorphic distribution D is always minimal in M, see [4]. This is also true for a CR-submanifold of a nearly Kaehler manifold, actually when D is integrable, each leaf of D is a minimal submanifold of both M and N, see [6]. We now consider the case of a CR-submanifold of a quasi-Kaehler manifold.

Proposition 4.1. *Let M be a CR-submanifold of a quasi-Kaehler manifold N. The holomorphic distribution D is minimal in M.*

Proof. Let $X \in \Gamma(D)$, $U \in \Gamma(D^\perp)$ then

$$\begin{aligned} g(U, \nabla_X X) &= g(U, \tilde{\nabla}_X X) = g(JU, J\tilde{\nabla}_X X) \\ &= g\left(JU, \tilde{\nabla}_X JX - (\tilde{\nabla}_X J)X\right) \end{aligned}$$

and hence

$$\begin{aligned} g(U, \nabla_X X + \nabla_{JX} JX) &= g(JU, \tilde{\nabla}_X JX - (\tilde{\nabla}_X J)X - \tilde{\nabla}_{JX} X - (\tilde{\nabla}_{JX} J)JX) \\ &= g(JU, \tilde{\nabla}_X JX - \tilde{\nabla}_{JX} X) \\ &= g(JU, [X, JX]) = 0. \end{aligned}$$

This implies that $\mathring{\mathrm{H}} = 0$ and so D is manimal in M.

As in the nearly Kaehler case, when D is integrable, each leaf of D is minimal in both M and N.

Theorem 4.2. *Let M be a CR-submanifold of a quasi-Kaeler manifold N. If the holomorphic distribution D is integrable then each leaf of D is a minimal submanifold in both M and N.*

Proof. By proposition 4.1. each leaf of D is minimal in M and hence $\tilde{\nabla}_X X + \tilde{\nabla}_{JX}$ has no component in $D^\perp$. For $Z \in \Gamma(TM)^\perp$ a similar computation shows that for $X \in \Gamma(D)$

$$g\left(Z, \tilde{\nabla}_X X + \tilde{\nabla}_{JX} JX\right) = g\left(JZ, [X, JX]\right) = 0,$$

this implies that each leaf of D is also a minimal submanifold of N.

It is well known that a complex submanifold of a Kaehler manifold is minimal, see [3, page 89]. When N is nearly Kaehler or quasi-Kaehler, N is not necessarily a complex manifold but nevertheless since an (almost) complex submanifold M of a quasi-Kaehler manifold N can be regarded as a CR-submanifold with $D_x = T_x M$, Theorem 4.2. yields the following result.

Corollary 4.3. *Every (almost) complex submanifold of a quasi-Kaehler manifold is minimal.*

References

[1] A. Bejancu, "CR-submanifolds of a Kaehler manifold I," *Proc. Amer. Math. Soc.*, 69 (1978), 135-142.

[2] A. Bejancu, Geometry of CR-submanifolds, Reidel Holland, 1986.

[3] B. Y. Chen, Geometry of submanifolds, Marcel Dekker, 1973.

[4] B. Y. Chen, "Cohomology of CR-submanifolds," *Ann. Fac. Sci. Toulouse, Math.*, 3 (1981), 167-172.

[5] A. Gray, "Minimal varieties and almost Hermitian submanifolds," *Michigan Math. J.*, 12 (1965), 273-287.

[6] S. H. Kon and Sin-Leng Tan, "CR-submanifolds of a nearly Kaehlerian Manifold," *Bull. Malaysian Math. Soc.* (Second Series), 14 (1991), 31-38.

Department of Mathematics, University of Malaya, 59100 Kuala Lumpur, Malaysia.