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ON CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS IN THE UNIT DISC I

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Abstract. The object of the present paper is to derive several interesting properties of the class $P_n(\alpha,\beta,\gamma)$ consisting of analytic and univalent functions with negative coefficients. Coefficient estimates, distortion theorems and closure theorems of functions in the class $P_n(\alpha,\beta,\gamma)$ are determined. Also radii of close-to-convexity, starlikeness and convexity and integral operators are determined.

1. Introduction

Let T denote the class of functions of the form

$$f(z)=a_1z-\sum_{k=2}^{\infty}a_kz^k \qquad (a_1>0;a_k\geq 0) \tag{1.1}$$

which are analytic and univalent in the unit disc $U=\{z:|z|<1\}$. For a function $f(z)$ in T, we define

$$D^0f(z)=f(z), \tag{1.2}$$

$$D^1f(z)=Df(z)=zf'(z), \tag{1.3}$$

and

$$D^nf(z)=D(D^{n-1}f(z)) \qquad (n\in N=\{1,2,\ldots\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [4]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to T is in the class $P_n(\alpha,\beta,\gamma)$ if and only if

$$\left|\frac{(D^nf(z))'-a_1}{\gamma(D^nf(z))'+(1-2\alpha\gamma)a_1}\right|<\beta \tag{1.5}$$

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where $0 \le \alpha < 1$, $0 < \beta \le 1$, $0 < \gamma \le 1$, and $n \in N_0 = N \cup \{0\}$.

We note that, by specializing the parameters α, β, γ, and n, we obtain the following subclasses studied by various authors:

(1) For $a_1 = 1$ and $n = 0$, $P_0(\alpha, \beta, \gamma) = P(\alpha, \beta, \gamma)$ (Owa [3]);

(2) For $a_1 = \gamma = 1$ and $n = 0$, $P_0(\alpha, \beta, 1) = P^*(\alpha, \beta)$ (Gupta and Jain [2]);

(3) For $a_1 = \beta = \gamma = 1$ and $n = 0$, $P_0(\alpha, 1, 1) = T^{**}(\alpha)$ (Sarangi and Uralegaddi [5] and Al-Amiri [1]);

(4) For $a_1 = \gamma = n = 1$, $P_1(\alpha, \beta, 1) = P_1(\alpha, \beta)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$\left| \frac{(zf'(z))' - 1}{(zf'(z))' + 1 - 2\alpha} \right| < \beta \tag{1.6}$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in P_n(\alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} k^{n+1}(1 + \beta\gamma)a_k \le \beta(1 + \gamma - 2\alpha\gamma)a_1. \tag{2.1}$$

The result is sharp.

Proof. Let $|z| = 1$. Then, we have

$$\begin{aligned} &|(D^n f(z))' - a_1| - \beta|\gamma(D^n f(z))' + (1 - 2\alpha\gamma)a_1| \\ &= \left| -\sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right| - \beta \left| (1 + \gamma - 2\alpha\gamma)a_1 - \gamma \sum_{k=2}^{\infty} k^{n+1} a_k z^{k-1} \right| \\ &\le \sum_{k=2}^{\infty} (1 + \beta\gamma) k^{n+1} a_k - \beta(1 + \gamma - 2\alpha\gamma)a_1 \le 0. \end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in P_n(\alpha, \beta, \gamma)$.

For the converse, assume that

$$\left| \frac{(D^n f(z))' - a_1}{\gamma(D^n f(z))' + (1 - 2\alpha\gamma)a_1} \right| = \left| \frac{-\sum\limits_{k=2}^{\infty} k^{n+1} a_k z^{k-1}}{(1 + \gamma - 2\alpha\gamma)a_1 - \gamma \sum\limits_{k=2}^{\infty} k^{n+1} a_k z^{k-1}} \right| < \beta.$$

Since $|\mathrm{Re}(z)| \le |z|$ for all z, we have

$$\mathrm{Re} \left\{ \frac{\sum\limits_{k=2}^{\infty} k^{n+1} a_k z^{k-1}}{(1 + \gamma - 2\alpha\gamma)a_1 - \gamma \sum\limits_{k=2}^{\infty} k^{n+1} a_k z^{k-1}} \right\} < \beta. \tag{2.2}$$

Choose values of z on the real axis so that $(D^n f(z))'$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{k=2}^{\infty} k^{n+1} a_k \leq \beta(1+\gamma-2\alpha\gamma)a_1 - \beta\gamma \sum_{k=2}^{\infty} k^{n+1} a_k.$$

This gives the required condition.

Finally, the function

$$f(z) = a_1 z - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)} z^k \qquad (k \geq 2) \tag{2.3}$$

is an extremal function for the theorem.

Corollary 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$. Then we have*

$$a_k \leq \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)} \qquad (k \geq 2;\, n \in N_0). \tag{2.4}$$

The equality in (2.4) *is attained for the function $f(z)$ given by* (2.3).

3. Some Properties of The Class $P_n(\alpha,\beta,\gamma)$

Theorem 2. *Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, and $n \in N_0$. Then*

$$P_n(\alpha,\beta,\gamma) = P_n\Big(\frac{1-\beta+2\alpha\beta\gamma}{1+\beta\gamma}, 1, 1\Big). \tag{3.1}$$

More generally, if $0 \leq \alpha' < 1$, $0 < \beta' \leq 1$, $0 < \gamma' \leq 1$, and $n \in N_0$, then

$$P_n(\alpha,\beta,\gamma) = P_n(\alpha',\beta',\gamma') \tag{3.2}$$

if and only if

$$\frac{\beta(1+\gamma-2\alpha\gamma)}{1+\beta\gamma} = \frac{\beta'(1+\gamma'-2\alpha'\gamma')}{1+\beta'\gamma'}. \tag{3.3}$$

Proof. First assume that the function $f(z)$ is in the class $P_n(\alpha,\beta,\gamma)$, and condition (3.3) holds. Then, by using the assertion (2.1) of Theorem 1, we readily have

$$\sum_{k=2}^{\infty} k^{n+1} a_k \leq \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{1+\beta\gamma} = \frac{\beta'(1+\gamma'-2\alpha'\gamma')a_1}{1+\beta'\gamma'},$$

which shows that $f(z) \in P_n(\alpha',\beta',\gamma')$, again with the aid of Theorem 1.

Reversing the above steps, we can similarly prove the other part of the equivalence (3.2) which, for $\beta' = \gamma' = 1$, immediately yields the special case (3.1).

Conversely, the assertion (3.2) can easily be shown to imply the condition (3.3), and the proof of Theorem 2 is thus completed.

Theorem 3. *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,$ *and* $n \in N_0$. *Then*

$$P_n(\alpha_2, \beta, \gamma) \subseteq P_n(\alpha_1, \beta, \gamma). \tag{3.4}$$

The proof of Theorem 3 uses Theorem 1 in a straight forward manner. The details are omitted.

Theorem 4. *Let* $0 \leq \alpha < 1,\ 0 < \beta_1 \leq \beta_2 \leq 1,\ 0 < \gamma \leq 1,$ *and* $n \in N_0$. *Then*

$$P_n(\alpha, \beta_1, \gamma) \subseteq P_n(\alpha, \beta_2, \gamma). \tag{3.5}$$

Proof. By using Theorem 2, we obtain

$$P_n(\alpha, \beta_1, \gamma) = P_n(\frac{1 - \beta_1 + 2\alpha\beta_1\gamma}{1 + \beta_1\gamma}, 1, 1) \tag{3.6}$$

and

$$P_n(\alpha, \beta_2, \gamma) = P_n(\frac{1 - \beta_2 + 2\alpha\beta_2\gamma}{1 + \beta_2\gamma}, 1, 1). \tag{3.7}$$

Furthermore

$$0 \leq \frac{1 - \beta_2 + 2\alpha\beta_2\gamma}{1 + \beta_2\gamma} \leq \frac{1 - \beta_1 + 2\alpha\beta_1\gamma}{1 + \beta_1\gamma} < 1 \tag{3.8}$$

for $0 \leq \alpha < 1$, $0 < \beta_1 \leq \beta_2 \leq 1$, and $0 < \gamma \leq 1$.
Consequently, by using Theorem 3, we arrive at our assertion (3.5).

Corollary 2. *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1,\ 0 < \beta_1 \leq \beta_2 \leq 1,\ 0 < \gamma \leq 1,$ *and* $n \in N_0$. *Then*

$$P_n(\alpha_2, \beta_1, \gamma) \subseteq P_n(\alpha_1, \beta_1, \gamma) \subseteq P_n(\alpha_1, \beta_2, \gamma).$$

Corollary 3. $P_{n+1}(\alpha, \beta, \gamma) \subset P_n(\alpha, \beta, \gamma)$ *for* $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,$ *and* $n \in N_0$.

Corollary 4. *Let* $\frac{1}{2} \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 < \gamma_1 \leq \gamma_2 \leq 1,$ *and* $n \in N_0$. *Then*

$$P_n(\alpha, \beta, \gamma_2) \subseteq P_n(\alpha, \beta, \gamma_1). \tag{3.9}$$

Proof. Let the function $f(z)$ defined by (1.1) be in the class $P_n(\alpha, \beta, \gamma_2)$. Then, by using Theorem 1,

$$\sum_{k=2}^{\infty} k^{n+1}(1 + \beta\gamma_1)a_k \leq \sum_{k=2}^{\infty} k^{n+1}(1 + \beta\gamma_2)a_k$$
$$\leq \beta(1 + \gamma_2 - 2\alpha\gamma_2)a_1 \leq \beta(1 + \gamma_1 - 2\alpha\gamma_1)a_1.$$

Hence $f(z) \in P_n(\alpha, \beta, \gamma_1)$.

Corollary 5. *Let* $\frac{1}{2} \leq \alpha_1 \leq \alpha_2 < 1$, $0 < \beta \leq 1$, $0 < \gamma_1 \leq \gamma_2 \leq 1$, *and* $n \in N_0$. *Then*

$$P_n(\alpha_2, \beta, \gamma_2) \subseteq P_n(\alpha_1, \beta, \gamma_1). \tag{3.10}$$

Proof. Let the function $f(z)$ defined by (1.1) be in the class $P_n(\alpha_2, \beta, \gamma_2)$. Then, by using Theorem 1,

$$\begin{aligned}\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma_1)a_k &\leq \sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma_2)a_k \\ &\leq \beta(1+\gamma_2-2\alpha_2\gamma_2)a_1 \leq \beta(1+\gamma_1-2\alpha_1\gamma_1)a_1.\end{aligned}$$

Hence $f(z) \in P_n(\alpha_1, \beta, \gamma_1)$.

4. Distortion Theorem

Theorem 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $P_n(\alpha, \beta, \gamma)$. *Then we have*

$$\left|D^i f(z)\right| \geq a_1|z| - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)}|z|^2 \tag{4.1}$$

and

$$\left|D^i f(z)\right| \leq a_1|z| + \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)}|z|^2 \tag{4.2}$$

for $z \in U$, *where* $0 \leq i \leq n$. *The result is sharp.*

Proof. Note that $f(z) \in P_n(\alpha, \beta, \gamma)$ if and only if $D^i f(z) \in P_{n-i}(\alpha, \beta, \gamma)$, and that

$$D^i f(z) = a_1 z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{4.3}$$

Using Theorem 1, we know that

$$\begin{aligned}2^{n+1-i}(1+\beta\gamma)\sum_{k=2}^{\infty} k^i a_k &\leq \sum_{k=2}^{\infty}(1+\beta\gamma)k^{n+1}a_k \\ &\leq \beta(1+\gamma-2\alpha\gamma)a_1,\end{aligned} \tag{4.4}$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)}. \tag{4.5}$$

If follows from (4.3) and (4.5) that

$$\begin{aligned}\left|D^i f(z)\right| &\geq a_1|z| - |z|^2 \sum_{k=2}^{\infty} k^i a_k \\ &\geq a_1|z| - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)}|z|^2 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\left|D^i f(z)\right| &\leq a_1|z| + |z|^2 \sum_{k=2}^{\infty} k^i a_k \\ &\leq a_1|z| + \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)}|z|^2. \end{aligned} \tag{4.7}$$

Finally, we note that the equality in (4.1) and (4.2) are attained by the function

$$D^i f(z) = a_1 z - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1-i}(1+\beta\gamma)} z^2 \tag{4.8}$$

or by

$$f(z) = a_1 z - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1}(1+\beta\gamma)} z^2. \tag{4.9}$$

Corollary 6. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$. Then we have*

$$|f(z)| \geq a_1|z| - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1}(1+\beta\gamma)}|z|^2 \tag{4.10}$$

and

$$|f(z)| \leq a_1|z| + \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n+1}(1+\beta\gamma)}|z|^2 \tag{4.11}$$

for $z \in U$. The result is sharp for the function $f(z)$ given by (4.9).

Proof. Taking $i = 0$ in Theorem 5, we can easily show (4.10) and (4.11).

Corollary 7. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$. Then we have*

$$|f'(z)| \geq a_1 - \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n}(1+\beta\gamma)} |z| \tag{4.12}$$

and

$$|f'(z)| \leq a_1 + \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{2^{n}(1+\beta\gamma)} |z| \tag{4.13}$$

for $z \in U$. The result is sharp for the function $f(z)$ given by (4.9).

Proof. Note that $D^1 f(z) = zf'(z)$. Hence, taking $i = 1$ in Theorem 5, we have the corollary.

Corollary 8. *Let the function $f(z)$ defined by (1.1) be in the class $P_n(\alpha, \beta, \gamma)$. Then $f(z)$ is included in a disc with its center at the origin and radius R_1 given by*

$$R_1 = a_1 \frac{2^{n+1}(1+\beta\gamma) + \beta(1+\gamma-2\alpha\gamma)}{2^{n+1}(1+\beta\gamma)}. \tag{4.14}$$

Further, $f'(z)$ is included in a disc with its center at the origin and radius R_2 given by

$$R_2 = a_1 \frac{2^{n}(1+\beta\gamma) + \beta(1+\gamma-2\alpha\gamma)}{2^{n}(1+\beta\gamma)}. \tag{4.15}$$

The result is sharp with the extremal function $f(z)$ given by (4.9).

5. Closure Theorems

Let the functions $f_j(z)$ $(j = 1, 2, \ldots, m)$ be defined by

$$f_j(z) = a_{1,j} z - \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (a_{1,j} > 0; a_{k,j} \geq 0) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $P_n(\alpha, \beta, \gamma)$.

Theorem 6. *Let the functions $f_j(z)$ $(j = 1, 2, \ldots, m)$ defined by (5.1) be in the class $P_n(\alpha, \beta, \gamma)$. Then the function $h(z)$ defined by*

$$h(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k \tag{5.2}$$

also belongs to the class $P_n(\alpha, \beta, \gamma)$, where

$$b_1 = \frac{1}{m} \sum_{j=1}^{m} a_{1,j} \quad \text{and} \quad b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j}. \tag{5.3}$$

Proof. Since $f_j(z) \in P_n(\alpha, \beta, \gamma)$, it follows from Theorem 1, that

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma) a_{k,j} \leq \beta(1+\gamma-2\alpha\gamma) a_{1,j} \quad (j = 1, 2, \ldots, m). \tag{5.4}$$

Therefore

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)b_k = \sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)\Big[\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\Big]$$
$$\leq \beta(1+\gamma-2\alpha\gamma)\Big[\frac{1}{m}\sum_{j=1}^{m} a_{1,j}\Big].$$

Hence we have

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)b_k \leq \beta(1+\gamma-2\alpha\gamma)b_1$$

which implies that $h(z) \in P_n(\alpha,\beta,\gamma)$.

Theorem 7. *Let the functions $f_j(z)$ defined by (5.1) be in the class $P_n(\alpha_j,\beta_j,\gamma_j)$ $(\frac{1}{2} \leq \alpha_j < 1,\ 0 < \beta_j \leq 1,\ 0 < \gamma_j \leq 1,\ n \in N_0)$ for each $j = 1,2,\ldots,m$. Then the function $h(z)$ defined by*

$$h(z) = \frac{1}{m}\sum_{j=1}^{m} a_{1,j}z - \frac{1}{m}\sum_{k=2}^{\infty}\Big[\sum_{j=1}^{m} a_{k,j}\Big]z^k \tag{5.5}$$

is in the class $P_n(\alpha,\beta,\gamma)$, where

$$\alpha = \min_{1\leq j\leq m}\{\alpha_j\},\ \beta = \max_{1\leq j\leq m}\{\beta_j\},\ \textit{and}\ \gamma = \min_{1\leq j\leq m}\{\gamma_j\}. \tag{5.6}$$

Proof. Since $f_j(z) \in P_n(\alpha_j,\beta_j,\gamma_j)$ for each $j = 1,2,\ldots,m$, we observe that

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta_j\gamma_j)a_{k,j} \leq \beta_j(1+\gamma_j-2\alpha_j\gamma_j)a_{1,j} \tag{5.7}$$

with the aid of Theorem 1. Therefore

$$\sum_{k=2}^{\infty} k^{n+1}\Big[\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\Big] = \frac{1}{m}\sum_{j=1}^{m}\sum_{k=2}^{\infty} k^{n+1}a_{k,j}$$
$$\leq \frac{1}{m}\sum_{j=1}^{m}\frac{\beta_j(1+\gamma_j-2\alpha_j\gamma_j)}{(1+\beta_j\gamma_j)}a_{1,j} \leq \frac{\beta(1+\gamma-2\alpha\gamma)}{(1+\beta\gamma)}\Big[\frac{1}{m}\sum_{j=1}^{m} a_{1,j}\Big].$$

Thus

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)\Big[\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\Big] \leq \beta(1+\gamma-2\alpha\gamma)\Big[\frac{1}{m}\sum_{j=1}^{m} a_{1,j}\Big], \tag{5.8}$$

which shows that $h(z) \in P_n(\alpha,\beta,\gamma)$, where α, β, and γ are given by (5.6).

Theorem 8. *The class $P_n(\alpha,\beta,\gamma)$ is closed under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1) be in the class $P_n(\alpha,\beta,\gamma)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=\lambda f_1(z)+(1-\lambda)f_2(z) \qquad (0\le\lambda\le 1) \tag{5.9}$$

is in the class $P_n(\alpha,\beta,\gamma)$. Since, for $0\le\lambda\le 1$,

$$h(z)=[\lambda a_{1,1}+(1-\lambda)a_{1,2}]z-\sum_{k=2}^{\infty}[\lambda a_{k,1}+(1-\lambda)a_{k,2}]z^k, \tag{5.10}$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty}k^{n+1}(1+\beta\gamma)[\lambda a_{k,1}+(1-\lambda)a_{k,2}]\le\beta(1+\gamma-2\alpha\gamma)[\lambda a_{1,1}+(1-\lambda)a_{1,2}] \tag{5.11}$$

which implies that $h(z)\in P_n(\alpha,\beta,\gamma)$.

As a consequence of Theorem 8, there exists the extreme points of the class $P_n(\alpha,\beta,\gamma)$.

Theorem 9. *Let $f_1(z)=a_1z$ and*

$$f_k(z)=a_1z-\frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)}z^k \qquad (k\ge 2) \tag{5.12}$$

for $0\le\alpha<1$, $0<\beta\le 1$, $0<\gamma\le 1$, and $n\in N_0$. Then $f(z)$ is in the class $P_n(\alpha,\beta,\gamma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z) \tag{5.13}$$

where $\lambda_k\ge 0$ $(k\ge 1)$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z)=a_1z-\sum_{k=2}^{\infty}\frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)}\lambda_k z^k. \tag{5.14}$$

Then we get

$$\sum_{k=2}^{\infty}\frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}\cdot\frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)}\lambda_k=\sum_{k=2}^{\infty}\lambda_k=1-\lambda_1\le 1. \tag{5.15}$$

By virtue of Theorem 1, this shows that $f(z) \in P_n(\alpha,\beta,\gamma)$.

On the other hand, suppose that the function $f(z)$ defined by (1.1) is in the class $P_n(\alpha,\beta,\gamma)$. Again, by using Theorem 1, we can show that

$$a_k \le \frac{\beta(1+\gamma-2\alpha\gamma)a_1}{k^{n+1}(1+\beta\gamma)} \qquad (k \ge 2). \tag{5.16}$$

Setting

$$\lambda_k = \frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}a_k \qquad (k \ge 2), \tag{5.17}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k. \tag{5.18}$$

Hence, we can see that $f(z)$ can be expressed in the form (5.13). This completes the proof of Theorem 9.

Corollary 9. *The extreme points of the class $P_n(\alpha,\beta,\gamma)$ are the functions $f_k(z)$ $(k \ge 1)$ given by Theorem 9.*

6. Radii of close-to-convexity, starlikeness and convexity

Theorem 10. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$, then $f(z)$ is close-to-convex of order δ $(0 \le \delta < 1)$ in $|z| < r_1(n,\alpha,\beta,\gamma,\delta)$, where*

$$r_1(n,\alpha,\beta,\gamma,\delta) = \inf_k \left\{ \frac{(a_1-\delta)k^n(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1} \right\}^{\frac{1}{k-1}} \qquad (k \ge 2). \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ given by (2.3).

Proof. We must show that $|f'(z) - a_1| \le a_1 - \delta$ for $|z| < \gamma_1(n,\alpha,\beta,\gamma,\delta)$. We have

$$|f'(z) - a_1| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - a_1| \le a_1 - \delta$ if

$$\sum_{k=2}^{\infty} (\frac{k}{a_1-\delta}) a_k |z|^{k-1} \le 1. \tag{6.2}$$

According to Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1} a_k \le 1. \tag{6.3}$$

Hence (6.2) will be true if

$$\frac{k|z|^{k-1}}{(a_1-\delta)} \le \frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}$$

or if

$$|z| \le \left\{\frac{(a_1-\delta)k^n(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.4}$$

The theorem follows easily from (6.4).

Theorem 11. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$, then $f(z)$ is starlike of order δ $(0 \le \delta < 1)$ in $|z| < r_2(n,\alpha,\beta,\gamma,\delta)$, where*

$$r_2(n,\alpha,\beta,\gamma,\delta) = \inf_k \left\{\frac{(1-\delta)k^{n+1}(1+\beta\gamma)}{(k-\delta)\beta(1+\gamma-2\alpha\gamma)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.5}$$

The result is sharp with the extremal function $f(z)$ given by (2.3).

Proof. It is sufficient to show that $|\frac{zf'(z)}{f(z)} - 1| \le 1-\delta$ for $|z| < r_2(n,\alpha,\beta,\gamma,\delta)$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{a_1 - \sum\limits_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $|\frac{zf'(z)}{f(z)} - 1| \le 1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{(k-\delta)a_k|z|^{k-1}}{(1-\delta)a_1} \le 1. \tag{6.6}$$

Hence, by using (6.3), (6.6) will be true if

$$\frac{(k-\delta)|z|^{k-1}}{(1-\delta)a_1} \le \frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}$$

or if

$$|z| \le \left\{\frac{(1-\delta)k^{n+1}(1+\beta\gamma)}{(k-\delta)\beta(1+\gamma-2\alpha\gamma)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.7}$$

The theorem follows easily from (6.7).

Corollary 10. *Let the function $f(z)$ defined by* (1.1) *be in the class $P_n(\alpha,\beta,\gamma)$, then $f(z)$ is convex of order δ $(0 \le \delta < 1)$ in $|z| < r_3(n,\alpha,\beta,\gamma,\delta)$, where*

$$r_3(n,\alpha,\beta,\gamma,\delta) = \inf_k \left\{\frac{(1-\delta)k^{n}(1+\beta\gamma)}{(k-\delta)\beta(1+\gamma-2\alpha\gamma)}\right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.8}$$

The result is sharp with the extremal function $f(z)$ given by (2.3).

7. Integral Operators

Theorem 12. *Let the function $f(z)$ defined by (1.1) be in the class $P_n(\alpha,\beta,\gamma)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{7.1}$$

also belongs to the class $P_n(\alpha,\beta,\gamma)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = a_1 z - \sum_{k=2}^{\infty} b_k z^k, \tag{7.2}$$

where

$$b_k = \Big[\frac{c+1}{c+k}\Big]a_k. \tag{7.3}$$

Therefore,

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)b_k = \sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)\Big[\frac{c+1}{c+k}\Big]a_k$$
$$\leq \sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)a_k \leq \beta(1+\gamma-2\alpha\gamma)a_1, \tag{7.4}$$

since $f(z)\in P_n(\alpha,\beta,\gamma)$. Hence, by Theorem 1, $F(z)\in P_n(\alpha,\beta,\gamma)$.

Theorem 13. *Let c be a real number such that $c > -1$. If $F(z)\in P_n(\alpha,\beta,\gamma)$, then the function defined by (7.1) is univalent in $|z| < r^*$, where*

$$r^* = \inf_k\Big[\frac{k^n(1+\beta\gamma)(c+1)}{\beta(1+\gamma-2\alpha\gamma)(c+k)}\Big]^{\frac{1}{k-1}} \quad (k\geq 2). \tag{7.5}$$

The result is sharp.

Proof. Let $F(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$ $(a_1 > 0; a_k \geq 0)$. It follows from (7.1) that

$$f(z) = \frac{z^{1-c}[z^cF(z)]'}{(c+1)} \qquad (c > -1)$$
$$= a_1 z - \sum_{k=2}^{\infty}\Big(\frac{c+k}{c+1}\Big)a_k z^k. \tag{7.6}$$

In order to obtain the required result it suffices to show that

$$|f'(z) - a_1| < a_1 \quad \text{in} \quad |z| < r^*.$$

Now

$$|f'(z) - a_1| \le \sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1}.$$

Thus $|f'(z) - a_1| < a_1$ if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)a_1} a_k |z|^{k-1} < 1. \tag{7.7}$$

Hence by using (6.3), (7.7) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{(c+1)a_1} < \frac{k^{n+1}(1+\beta\gamma)}{\beta(1+\gamma-2\alpha\gamma)a_1}$$

or if

$$|z| < \left[\frac{k^n(1+\beta\gamma)(c+1)}{\beta(1+\gamma-2\alpha\gamma)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.8}$$

Therefore $f(z)$ is univalent in $|z| < r^*$. Sharpness follows if we take

$$f(z) = a_1 z - \frac{\beta(1+\gamma-2\alpha\gamma)a_1(c+k)}{k^n(1+\beta\gamma)(c+1)} z^k \qquad (k \ge 2). \tag{7.9}$$

8. Linear Combinations of Functions in $P_n(\alpha, \beta, \gamma)$

Theorem 14. *Let the functions $f_j(z)$ defined by (5.1) be in the class $P_n(\alpha, \beta, \gamma)$ for every $j = 1, 2, \ldots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z) \qquad (c_j > 0) \tag{8.1}$$

is in the same class $P_n(\alpha, \beta, \gamma)$.

Proof. By the definition of $h(z)$, we have

$$h(z) = \left[\sum_{j=1}^{m} c_j a_{1,j}\right] z - \sum_{k=2}^{\infty}\left[\sum_{j=1}^{m} c_j a_{k,j}\right] z^k. \tag{8.2}$$

Further, since $f_j(z)$ are in $P_n(\alpha, \beta, \gamma)$ for every $j = 1, 2, \ldots, m$, we get

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)a_{k,j} \le \beta(1+\gamma-2\alpha\gamma)a_{1,j}. \tag{8.3}$$

for every $j = 1, 2, \ldots, m$. Hence we can see that

$$\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)\Big[\sum_{j=1}^{m} c_j a_{k,j}\Big] = \sum_{j=1}^{m} c_j \Big[\sum_{k=2}^{\infty} k^{n+1}(1+\beta\gamma)a_{k,j}\Big] \le \beta(1+\gamma-2\alpha\gamma)\sum_{j=1}^{m} c_j a_{1,j}.$$

This proves that $h(z) \in P_n(\alpha, \beta, \gamma)$.

Remark. Owa [3] considered the class $P_0(\alpha,\beta,\gamma) = P(\alpha,\beta,\gamma)$ of functions $f(z)$ defined by (1.1) and satisfying

$$\left|\frac{f'(z)-1}{\gamma f'(z)+(1-2\alpha\gamma)}\right| < \beta \tag{i}$$

where $0 \le \alpha < 1$, $0 < \beta \le 1$ and $0 < \gamma \le 1$.

One can easily verify that the condition (i) is equivalent to

$$f'(z) = \frac{1+\beta(1-2\alpha\gamma)w(z)}{1-\beta\gamma w(z)}, \qquad z \in U, \tag{ii}$$

where $w(z)$ is a function analytic in U and satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. Since $f'(z) = a_1 - \sum_{k=2}^{\infty} k a_k z^{k-1}$, it follows that the constant term in the Taylor expansion of both sides of (ii) is not the same except when $a_1 = 1$. It seems, therefore, that the class $P(\alpha,\beta,\gamma)$ has not been defined by Owa [3] in proper way. In fact, the correct form of (i) must be

$$\left|\frac{f'(z)-a_1}{\gamma f'(z)+(1-2\alpha\gamma)a_1}\right| < \beta, \qquad z \in U. \tag{iii}$$

Consequently, the correct form of (ii) is

$$f'(z) = a_1\frac{1+\beta(1-2\alpha\gamma)w(z)}{1-\beta\gamma w(z)}, \qquad z \in U. \tag{iv}$$

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