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COINCIDENCE POINT THEOREMS FOR MULTI-VALUED AND SINGLE-VALUED MAPPINGS IN MENGER PM-SPACES

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Abstract. In this paper, coincidence point theorems for compatible mappings of multi-valued and single-valued mappings in Menger PM-spaces are obtained. The presented results in this paper generalize and improve the corresponding results of S. S. Chang et al., O. Hadžić and others.

Throughout this paper, let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$, Z^+ be the set of all positive integers and $\mathcal{D}$ be the set of all distribution finctions on R.

If $(X, \mathcal{F}, T)$ is a Menger probabilistic metric space (shortly, a Menger PM-space) with the continuous t-norm T, then $(X, \mathcal{F}, T)$ is a Hausdorff space [6] in the topology τ induced by the faimily

$$\{U_x(\epsilon, \lambda) : x \in X,\ \epsilon > 0,\ \lambda \in (0,1)\}$$

of neighborhoods $U_x(\epsilon, \lambda)$, where

$$U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda\}.$$

Refer to [1], [2] and [6]-[8] for other details on PM-spaces.

Definition 1. Let $(X, \mathcal{F}, T)$ be a Menger PM-space with the continuous t-norm T and A be a nonempty subset of X. $D_A(t) = \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s)$ is called the *probabilistic diameter* of A. Specifically, the set A is said to be *probabilistically bounded* if

$$\lim_{t \to \infty} D_A(t) = 1.$$

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From [2], we know that

(1) if A is a probabilistically bounded subset of X, then $D_A(t)$ is a distribution function,

(2) if A and B are probabilistically bounded subsets of X, then $A \cup B$ is also a probabilistically bounded set.

Let $(X, \mathcal{F}, T)$ be a Menger PM-space with the continuous t-norm T and $CB(X)$ be the family of nonempty probabilistically bounded and closed subset of X. We define a mapping $\widetilde{\mathcal{F}} : CB(X) \times CB(X) \to \mathcal{D}$ as follows (we denote $\widetilde{\mathcal{F}}(A, B)$ by $\widetilde{F}_{A,B}$ and the value $\widetilde{F}(A, B)$ at $t \in R$ by $\widetilde{F}_{A,B}(t)$):

$$\widetilde{F}_{A,B}(t) = \sup_{s<t} T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)), \quad A, B \in CB(X).$$

The mapping $\widetilde{\mathcal{F}}$ is called the *Menger-Hausdorff metric* induced by $\mathcal{F}$.

Proposition 1. [2] *$(CB(X), \widetilde{F}, T)$ be a Menger PM-space, i.e., $\widetilde{F}$ is a mapping satisfying the following conditions*:

(1) $\widetilde{F}_{A,B}(t) = 1$ *for all* $t > 0$ *if and only if* $A = B$,

(2) $\widetilde{F}_{A,B}(0) = 0$,

(3) $\widetilde{F}_{A,B}(t) = \widetilde{F}_{B,A}(t)$,

(4) $\widetilde{F}_{A,B}(t_1 + t_2) \geq T(\widetilde{F}_{A,C}(t_1), \widetilde{F}_{C,B}(t_2))$ *for all* $A, B \in CB(X)$ *and* $t_1, t_2 > 0$.

Definition 2. Let $A \in CB(X)$ and $x \in X$. The *probabilistic distance* between x and A is the function defined by

$$F_{x,A}(t) = \sup_{s<t} \sup_{y\in A} F_{x,y}(s), \qquad t > 0.$$

Proposition 2. [2] *Let $A, B \in CB(X)$ and $x, y \in X$. Then we have the following*:

(1) $F_{x,A}(t) = 1$ *for all* $t > 0$ *if and only if* $x \in A$,

(2) $F_{x,A}(t_1 + t_2) \geq T(F_{x,y}(t_1), F_{y,A}(t_2))$ *for all* $t_1, t_2 > 0$,

(3) $F_{x,B}(t) \geq \widetilde{F}_{A,B}(t)$ *for all* $t > 0$.

Remark 1. If we define $F_{x,A}(t) = \sup_{y\in A} F_{x,y}(t)$, then, since we have

$$\sup_{s<t} \sup_{y\in A} F_{x,y}(s) = \sup_{y\in A} F_{x,y}(t),$$

$F_{x,A}(t)$ is a left-continuous at t and the properties (1), (2), (3) of Proposition 2 are also satisfied.

Definition 3. Let $\phi : R^+ \to R^+$ be a strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. Define a function $\psi : R^+ \to R^+$ by

$$\text{(A)} \qquad \psi(t) = \begin{cases} 0 & t = 0, \\ \inf\{s > 0 : \phi(s) > t\}, & t > 0. \end{cases}$$

It is easy to prove that $\psi : R^+ \to R^+$ is a continuous and nondecreasing function.

Definition 4. We say that a function $\phi : R^+ \to R^+$ satisfies the condition (Φ) if it is a strictly increasing and left-continuous function such that $\phi(0) = 0$, $\lim_{t\to+\infty} \phi(t) = +\infty$ and $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^n(t)$ is the n-th iteration of $\phi(t)$.

Lemma 3. [3] *Let $\phi : R^+ \to R^+$ satisfy the condition (Φ) and let ψ be defined by (A). Then we have the following:*
(1) $\phi(t) < t$ *for all* $t > 0$,
(2) $\phi(\psi(t)) \le t$ *and* $\psi(\phi(t)) = t$ *for all* $t \ge 0$,
(3) $\psi(t) \ge t$ *for all* $t \ge 0$,
(4) $\lim_{n\to\infty} \psi^n(t) = +\infty$ *for all* $t > 0$.

Definition 5. A t-norm T is said to be of h-type if the family $\{T^m(t)\}_{m=1}^{\infty}$ of the functions $T^m(t) = T(t, T^{m-1}(t))$, $m = 1, 2, \ldots$, $T^0(t) = t$, $t \in [0, 1]$, is equicontinuous at 1.

Lemma 4. [3] *Let $(X, \mathcal{F}, T)$ be a Menger PM-space, where T is a t-norm of h-type. If a sequence $\{x_n\}$ in X satisfies the following condition:*
For any $n \in Z^+$ and $t > 0$,

$$F_{x_n,x_{n+1}}(t) \ge F_{x_0,x_1}(\psi^n(t)), \tag{B}$$

where ϕ is a function satisfying the condition (Φ) and ψ is defined by (A), *then the sequence $\{x_n\}$ is a Cauchy sequence in X.*

Now we introduce the concepts of weakly compatible mappings for multi-valued and single-valued mappings in the setting of PM-spaces, which is motivated by the concept of compatible mappings for single-valued mappings in metric spaces [5].

Definition 6. Let f be a mapping from X into itself and S be a multi-valued mappings X into $CB(X)$.
(1) The mappings f and S are said to be *commuting* if $fSx \in CB(X)$ and $fSx = Sfx$ for all $x \in X$,
(2) The mappings f and S are said to be *weakly compatible* if $fSx = CB(X)$ and

$$\lim_{n\to\infty} \widetilde{F}_{fSx_{n-1},Sfx_{n-1}}(t) = 1, \qquad t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} F_{fx_n,Sx_{n-1}}(t) = 1, \qquad t > 0.$$

Remark 2. From Definition 6, any commuting mappings are weakly compatible but the converse is not true (cf. [5]).

Now we are in a position to give our main results:

Theorem 5. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with the continuous t-norm T of h-type. Let f be a continuous self-mapping of X and S be a continuous multi-valued mapping from X into $CB(X)$ satisfying the following conditions:*
(C) $S(X) \subset f(X)$,
(D) *For any $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that*

$$F_{u,v}(\phi(t)) \geq \min\{F_{fx,fy}(t), F_{fx,Sx}(t), F_{fy,Sy}(t)\}$$

for all $t > 0$, where the function ϕ satisfies the condition (Φ).

If f and S are weakly compatible, then there exists a point $z \in X$ such that $fz \in Sz$, that is, the point z is a coincidence point of f and S.

Proof. First, we shall show that we can find a sequence $\{x_n\}_{n \in Z^+}$ in X such that

$$F_{fx_n, fx_{n+1}}(t) \geq F_{fx_0, fx_1}(\psi^n(t))$$

for all $t > 0$ and $n \in Z^+$.

From (C), for arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $fx_1 \in Sx_0$. By Lemma 3 and (D), there exists a point $x_2 \in X$ such that $fx_2 \in Sx_1$ and

$$\text{(E)} \qquad \begin{aligned} F_{fx_1,fx_2}(t) &\geq F_{fx_1,fx_2}(\phi(\psi(t))) \\ &\geq \min\{F_{fx_0,fx_1}(\psi(t)), F_{fx_0,Sx_0}(\psi(t)), F_{fx_1,Sx_1}(\psi(t))\} \\ &\geq \min\{F_{fx_0,fx_1}(\psi(t)), F_{fx_1,fx_2}(\psi(t))\} \end{aligned}$$

for all $t > 0$, where $\psi(t)$ is defined by (A). Using (E) repeatedly, we have

$$\text{(F)} \qquad \begin{aligned} F_{fx_1,fx_2}(t) &\geq \min\{F_{fx_0,fx_1}(\psi(t)), F_{fx_0,fx_1}(\psi^2(t)), F_{fx_1,fx_2}(\psi^2(t))\} \\ &= \min\{F_{fx_0,fx_1}(\psi(t)), F_{fx_1,fx_2}(\psi^2(t))\} \\ &\geq \cdots \\ &\geq \min\{F_{fx_0,fx_1}(\psi(t)), F_{fx_1,fx_2}(\psi^n(t))\}. \end{aligned}$$

Letting $n \to \infty$ in (F), we have

$$F_{fx_1,fx_2}(t) \geq F_{fx_0,fx_1}(\psi(t)), \qquad t > 0.$$

By repeating the above procedure, we can define a sequence $\{x_n\}$ in X such that $fx_{n+1} \in Sx_n$ and

$$F_{fx_n,fx_{n+1}}(t) \geq F_{fx_{n-1},fx_n}(\psi(t)), \qquad t > 0.$$

Thus, for any $n \in Z^+$ and $t > 0$, we have

$$\text{(G)} \qquad F_{fx_n,fx_{n+1}}(t) \geq F_{fx_{n-1},fx_n}(\psi(t)) \geq \cdots \geq F_{fx_0,fx_1}(\psi^n(t))$$

and so, by Lemma 4, $\{fx_n\}$ is a Cauchy sequence in X. Since $(X,\mathcal{F},T)$ is complete, we can assume that $\lim_{n\to\infty} fx_n = z \in X$. Since f and S are weakly compatible and the sequence $\{x_n\}$ in X is such that $fx_n \in Sx_{n-1}$, by Proposition 2, we have

$$\lim_{n\to\infty} F_{fx_n,Sx_{n-1}}(t) = 1, \qquad t > 0,$$

and so

$$\lim_{n\to\infty} \widetilde{F}_{fSx_{n-1},Sfx_{n-1}}(t) = 1, \qquad t > 0.$$

On the other hand, from (3) of Proposition 2, it follows that

$$\text{(H)} \qquad F_{ffx_n,Sf_{n-1}}(t) \geq \widetilde{F}_{fSx_{n-1},Sfx_{n-1}}(t), \qquad t > 0.$$

Letting $n \to \infty$ in (H), we have

$$1 = \lim_{n\to\infty} F_{ffx_n,Sf_{n-1}}(t) = F_{fz,Sz}(t)$$

and so, by Proposition 2, $fz \in Sz$, that is, z is a coincidence point of f and S. This completes the proof.

In Theorem 5, by strengthening the weak compatibility of f and S by the commutitivity of f and S, we have the following:

Theorem 6. *Let $(X,\mathcal{F},T)$ be a complete Menger PM-space with the continuous t-norm T of h-type. Let f and S be as in Theorem 5 satisfying the conditions (C) and (D). If f and S commute, then there exists a point $z \in X$ such that $fz \in Sz$.*

Proof. Following the same argument as in Theorem 5, we can assume that $fx_n \to z \in X$.

Next, we prove that z is a coincidence point of f and S. In fact, for any $t > 0$ and $\epsilon \in (0,t)$, from (G) and the commutativity of f and S, we have

$$\text{(I)} \qquad \begin{aligned} &F_{ffx_{n+1},Sz}(t-\epsilon) \\ &\geq F_{ffx_{n+1},Sz}(\phi(\psi(t-\epsilon))) \\ &= \sup_{fy\in Sz} F_{ffx_{n+1},fy}(\phi(\psi(t-\epsilon))) \\ &\geq \min\{F_{ffx_n,fz}(\psi(t-\epsilon)), F_{ffx_n,ffx_{n+1}}(\psi(t-\epsilon)), F_{fz,Sz}(\psi(t-\epsilon))\} \\ &\geq \min\{F_{ffx_n,fz}(\psi(t-\epsilon)), F_{ffx_0,ffx_1}(\psi^{n+1}(t-\epsilon)), F_{fz,Sz}(\psi(t-\epsilon))\}. \end{aligned}$$

If $F_{fz,Sz}(\psi(t-\epsilon)) = 1$, then we have

$$\text{(J)} \qquad F_{ffx_{n+1},Sz}(t-\epsilon) \geq \min\{F_{ffx_n,fz}(\psi(t-\epsilon)), F_{ffx_0,ffx_1}(\psi^{n+1}(t-\epsilon))\}.$$

Letting $n \to \infty$ in (J), we have $F_{fz,Sz_0}(t-\epsilon) \geq 1$. Since $\epsilon \in (0,t)$ is arbitrary, we have $F_{fz,Sz}(t) = 1$ for all $t > 0$, that is, $fz \in Sz$.

If $F_{fz,Sz}(\psi(t-\epsilon)) < 1$, then, letting $n \to \infty$ in (I), we have

(K) $$F_{fz,Sz}(t-\epsilon) \geq F_{fz,Sz}(\psi(t)).$$

By using (2) of Proposition 2, we have

(L) $$F_{fz,Sz}(t) \geq F_{fz,Sz}(t-\epsilon).$$

Thus, as $\epsilon \to t$, from the continuity of ψ and the left-continuity of $F_{fz,Sz}$, it follows that

$$F_{fz,Sz}(t) \geq F_{fz,Sz}(\psi(t)).$$

Taking this procedure repeatedly, we obtain

$$F_{fz,Sz}(t) \geq F_{fz,Sz}(\psi(t)) \geq \cdots \geq F_{fz,Sz}(\psi^n(t)).$$

Therefore, as $n \to \infty$, $F_{fz,Sz}(t) = 1$ for all $t > 0$, that is, $fz \in Sz$. This completes the proof.

Taking $f =$ the identity mapping on X in Theorem 6, we obtain the following:

Corollary 7. [3] *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with the continuous t-norm T of h-type. Let S be a multi-valued mappings from X into $CB(X)$ satisfy the condition:*
(M) *For any $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that*

$$F_{u,v}(\phi(t)) \geq \min\{F_{x,y}(t), F_{x,Sx}(t), F_{y,Sy}(t)\}$$

for all $t > 0$, where the function ϕ satisfies the condition (Φ). Then S has a fixed point in X, that is, there exists a point $z \in X$ such that $z \in Sz$.

Taking $\phi(t) = kt$, $0 < k < 1$, in Corollary 7, we have the following:

Corollary 8. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with the continuous t-norm T of h-type. Let S be a multi-valued mapping from X into $CB(X)$ satisfying the following condition:*
(N) *For any $x, y \in X$ and $u \in Sx$, there exists a point $v \in Sy$ such that*

$$F_{u,v}(kt) \geq \min\{F_{x,y}(t), F_{x,Sx}(t), F_{y,Sy}(t)\}$$

for all $t > 0$, where $k \in (0,1)$ is a constant. Then S has a fixed point in X.

Corollary 8 is a generalization of the following result of O. Hadžić ([4]):

Corollary 9. *Let $(X, \mathcal{F}, T)$ be a complete Menger PM-space with continuous t-norm T, M be a nonempty probabilistically bounded subset of X. Let S be a multi-valued mapping from M into $C(M)$, the family of nonempty compact subsets of X, satisfying the following condition:*
(O) *For any $x, y \in M$ and $u \in Sx$, there exists a point $v \in Sy$ such that*

$$F_{u,v}(kt) \geq F_{x,y}(t)$$

for all $t > 0$, where $k \in (0,1)$ is a constant. Then S has a fixed point in M.

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