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ON THE CESÀRO SUMMABILITY OF ULTRASPHERICAL SERIES

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Abstract. In this paper the Cesàro summability of the ultraspherical series has been investigated which extend and generalize the results of Wang [5, 6] of Fourier series.

1. Let $f(\theta,\phi)$ be a function defined for the range $0\le\theta\le\pi$, $0\le\phi\le 2\pi$ on a sphere S. The ultraspherical series associated with this function is,

$$f(\theta,\phi)\sim\frac{1}{2\pi}\sum_{n=0}^{\infty}(n+\lambda)\int\int_S\frac{P_n^{(\lambda)}(\cos w)f(\theta',\phi')d\sigma'}{[\sin^2\theta'\sin^2(\phi-\phi')]^{1/2-\lambda}};\qquad \lambda>0 \tag{1.1}$$

where w is the spherical distance between the points (θ',ϕ'), i.e.

$$\cos w=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

and $d\sigma'=\sin\theta'\,d\theta'\,d\phi'$

The Laplace series is a particular case of the series of (1.1) for $\lambda=1/2$, while it reduces to the trigonometric series in the limit as $\lambda\to 0$, because

$$\lim_{\lambda\to 0}\ \frac{1}{\lambda}P_n^{(\lambda)}(\cos\theta)=\frac{2}{n}\cos n\theta,\qquad n\ge 1. \tag{1.2}$$

The ultraspherical polynomials $P_n^{(\lambda)}(x)$ are defined by the following expansion:

$$[1-2xz+z^2]^{-\lambda}=\sum_{n=0}^{\infty}z^nP_n^{(\lambda)}(x),\qquad \lambda>0. \tag{1.3}$$

We suppose throughout that the function

$$f(\theta',\phi')[\sin^2\theta'\sin^2(\phi-\phi')]^{\lambda-1/2} \tag{1.4}$$

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is integrable (L) over the whole surface of the unit sphere and following Kogbetliantz [2] we define the generalised mean value of $f(\theta, \phi)$ as follows:

$$f(w) = \frac{1}{2\pi(\sin w)^2} \int_{C_w} \frac{f(\theta', \phi')dS'}{[\sin^2 \theta' \sin^2(\phi - \phi')]^{1/2-\lambda}} \tag{1.5}$$

where the integral is taken along the small circle C_w, where ds' is element of the arc of C_w, whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is w.
We write,

$$\begin{aligned}
\phi(w) &= \Big[f(w) - \frac{A\Gamma(\lambda)}{\Gamma(1/2)\Gamma(1/2+\lambda)}\Big](\sin w)^{2\lambda};\\
\Phi_p(x) &= \frac{1}{\Gamma(p)} \int_0^x (x-t)^{p-1}\phi(t)dt;\\
\Phi_0(x) &= \phi(x);\\
\phi_p(x) &= \Gamma(p+1)x^{-p}\Phi_p(x), \qquad p \geq 0;
\end{aligned}$$

and

$$\Phi_p(x) = \frac{d}{dx}\Phi_{p+1}(x), \qquad -1 < p < 0.$$

The authors have obtained a theorem for Cesàro summability of the series (1.1) analogous to those of Izumi and Sonouchi [1]. The object of this paper is to extend and generalize the result of Wang [5 & 6] foFr the same series.
We prove the following:

Theorem. *If* $y > x > 0$ *and*

$$\Phi_x(t) = o(t^{y+2\lambda+1/\lambda}) \qquad for\ o < \lambda < 1.$$

then the series (1.1) *is summable* $(C, \alpha + \lambda)$ *at the point* (θ, ϕ), *where*

$$\alpha = \frac{my + x - y}{m + y - x}$$

and m *is a positive integer, such that* $m \geq x > m - 1$.

2. For the proof of the theorem we require the following Lemmas:

Lemma 1. *Let* $S_n^k(w)$ *denote the* n^{th} *Cesàro mean of order* k *of the series*

$$\sum (n+\lambda)P_n^{(\lambda)}(\cos w) \tag{2.1}$$

Then we have, for $\lambda > 0$ and $p \geq 0$,

$$S_n^p(w) = \frac{d^p(S_n^k(w))}{dw^p} = \begin{cases} O(n^{2\lambda+p+1}) & \text{for } 0 \leq w \leq \pi, k > 0; \\ O(\frac{n^{\lambda+p-k}}{w^{\lambda+k+1}}) + O(\frac{1}{nw^{2\lambda+p+2}}) & \text{for } 0 < w \leq a < \pi; \\ O(\frac{n^{\lambda+p-k}}{w^{k+\lambda+1}}) & \text{for } 0 < w \leq a < \pi \text{ and } \\ & \lambda + 1 + [p] \geq k \end{cases} \tag{2.2}$$

Lemma 2. *In order that the series (1.1) be summable (C, k), it is sufficient that the integral*

$$i = \int_0^\delta \phi(w) S_n^k(w) dw = o(1) \tag{2.3}$$

For $0 < \delta < \pi$ and for each $k > \lambda$.

Lemma 3. *For a non-integral*

$$\delta = m + \sigma, \qquad (0 < \sigma < 1),$$

We have

$$\int_0^\Delta \Phi_\delta(u) S_n^{(\delta)}(u) du = \Phi_{m+1}(\Delta) S_n^{(m)}(\Delta) \times \int_0^\Delta \Phi_m(t) S_n^{(m)}(t) dt. \tag{2.4}$$

Lemma 4. *If $0 \leq u \leq \frac{1}{n}$*

$$F(n, u) = O(n^{2\lambda+m+1} u^{m-x}) + O(n^{2\lambda+m+1} u^{m-x-1}) \tag{2.5}$$

where

$$F(n, u) = \Gamma \frac{1}{(m-x)} \int_u^{1/n} (t-u)^{m-x-1} S_n^{(m)}(t) dt \tag{2.6}$$

Lemmas 1, 2, 3 and 4 are due to Obrechkoff [3] and Singhai [4] respectively.

3. Proof of the Theorem

In order to prove the theorem, in view of (2.3) it is sufficient to show that

$$\lim_{n \to \infty} \int_0^\delta \phi(u) S_n^{\alpha+\lambda}(u) du = 0$$

under the condition of the theorem.

We have the following inequality:

$$x > \alpha > m - 1 \quad \text{(F. T. Wang)}$$

Also we have

$$x + 1 > m \geq x$$

Then

$$\begin{aligned} i &= \int_0^\delta \phi(w) S_n^k(w) dw \\ &= \Big[\sum_{p=1}^{m} (-1)^{p-1} \Phi_p(w) (\frac{d}{dw})^{p-1} S_n^{\alpha+\lambda}(w)\Big]_0^\delta + (-1)^m \int_0^\delta \Phi_m(t) S_n^{(m)}(t) dt. \\ &= I_1 + (-1)^m I_2 \end{aligned}$$

Since $\alpha > m - 1$

$$I_1 = o(1) \quad \text{as} \quad n \longrightarrow \infty. \tag{3.1}$$

We write

$$\begin{aligned} I_2 &= \int_0^\delta \Phi_m(t) S_n^{(m)}(t) dt \\ &= \int_0^{1/n} + \int_{1/n}^\delta = I_{2.1} + I_{2.2} \end{aligned}$$

$$\begin{aligned} I_{2.2} &= [\Phi_m(t) S_n^{(m-1)}(t)]_{1/n}^\delta - \int_{1/n}^\delta \Phi_{m-1}(t) S_n^{(m-1)}(t) dt \\ &= I_{2.2.1} - I_{2.2.2} \end{aligned} \tag{3.2}$$

we have

$$\begin{aligned} I_{2.2.1} &= o(1) + o(\frac{1}{n^{y+2\lambda+1/\lambda}}) O(\frac{n^{m-1-\alpha}}{n^{-\alpha-1-2\lambda}}) \\ &= o(1) \quad \text{as} \quad n \longrightarrow \infty \end{aligned}$$

Also if we write

$$\Phi^*(t) = \int_0^t |\Phi_{m-1}(u)| du$$

then

$$\begin{aligned} I_{2.2.2} &= O[n^{m-1-\alpha} \int_{1/n}^\delta \frac{|\Phi_{m-1}(t)|}{t^{\alpha+2\lambda+1}} dt] \\ &= O(n^{m-1-\alpha}) [\frac{\Phi^*(t)}{t^{\alpha+2\lambda+1}}]_{1/n}^\delta + O(n^{m-1-\alpha}) \int_{\frac{1}{n}}^\delta \frac{\Phi^*(t)}{t^{\alpha+2\lambda+2}} dt \\ &= O(n^{m-1-\alpha}) + O(n^{m-1-\alpha}) \circ (\frac{1}{n^{y+2\lambda+1/\lambda}}) \\ &\quad \times n^{\alpha+2\lambda+1} + O(n^{m-1-\alpha}) \int_{1/n}^\delta \frac{o(t^{y+2\lambda+1/\lambda})}{t^{\alpha+2\lambda+2}} dt \\ &= o(1) \quad \text{as} \quad n \longrightarrow \infty \end{aligned} \tag{3.3}$$

When x is not an integer.

$$\begin{aligned} I_{2.1} &= \int_0^{1/n} \Phi_m(t) S_n^{(m)}(t)dt \\ &= \int_0^{1/n} \Phi_x(u) F(n,u)du \end{aligned}$$

where $F(n,u) = \frac{1}{\Gamma(m-x)} \int_u^{1/n} (t-u)^{m-x-1} S_n^{(m)}(t)dt$
Now

$$\begin{aligned} I_{2.1} &= o[\int_0^{1/n} (y + 2\lambda + 1/\lambda) O(n^{2\lambda+m+1} u^{m-x}) du] \\ &\quad + o[\int_0^{1/n} u^{(y+2\lambda+1/\lambda)} O(n^{2\lambda+m+1} u^{m-x-1}) du] \\ &= o(1) \quad \text{as} \quad n \longrightarrow \infty. \end{aligned} \tag{3.4}$$

When $x = m$ is an integer.

$$\begin{aligned} I_{2.1} &= \int_0^{1/n} \Phi_x(t) S_n^{(x)}(t)dt \\ &= o[\int_0^{1/n} u^{y+2\lambda+1/\lambda} O(n^{2\lambda+x+1}) du] \\ &= o(1) \quad \text{as} \quad n \longrightarrow \infty. \end{aligned} \tag{3.5}$$

combining (3.1), (3.2), (3.3), (3.4) and (3.5) the result is proved.

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