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A FURTHER IMPROVEMENT OF JENSEN'S INEQUALITY

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1. Introduction

In the Theory of Inequalities, the famous Jensen's discrete inequality:

$$f(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

valid for every convex function $f : C \subseteq X \to \mathbf{R}$ (C is a convex subset of linear space X) and for every $x_i \in C$ and $p_i \ge 0$ $(i = 1, \cdots, n)$ with $P_n := \sum_{i=1}^{n} p_i > 0$ plays such an important role that many mathematicians have tried not only to establish (1) in a variety of ways but also to find different extensions, refinements and counterparts; see [3] and [7] where further references are given.

In the recent paper [10], J. E. Pečarić and the author have obtained the following refinement of (1):

Theorem A. *Let f, x_i, p_i be as above and $1 \le k \le n$. Then one has the inequalities:*

$$\begin{aligned} f(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i) &\le \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f(\frac{x_{i_1}+\cdots+x_{i_k}}{k}) \\ &\le \cdots \le \frac{1}{P_n^2}\sum_{i_1,i_2=1}^{n} p_{i_1}p_{i_2} f(\frac{x_{i_1}+x_{i_2}}{2}) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \end{aligned} \tag{2}$$

Another result of this type for weighted means was established by the author in [6]:

Theorem B. *Let f, x_i, p_i, k be as above and $q_j \ge 0$ $(j = 1, \cdots, k)$ with $Q_k := \sum_{j=1}^{k} q_j > 0$. Then one has the inequality:*

$$f(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i) \le \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j})$$

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$$\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{3}$$

For some applications of these results in Theory of Inequalities we refer to [4], [6] and [10].

The main purpose of this paper is to give an improvement of refinement (3) as follows. Some natural applications are also pointed out.

2. The main reults

The following theorem holds.

Theorem. *Let $f : C \subseteq X \to \mathbf{R}$ be a convex mapping on convex subset C of linear space X, $x_i \in C$, $p_i \geq 0$ $(i = 1, \cdots, n)$ with $P_n > 0$ and $q_j \geq 0$ $(j = 1, \cdots, k;$ $1 \leq k \leq n)$ with $Q_k > 0$. Then one has the inequalities*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{x_{i_1}+\ldots+x_{i_k}}{k}\right) \tag{4}$$

$$\leq \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{q_1 x_{i_1}+\ldots+q_k x_{i_k}}{Q_k}\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

Proof. We must prove only the second inequality.

Let consider the vectors

$$y_1 := \frac{1}{Q_k}(q_1 x_{i_1} + q_2 x_{i_2} + \ldots + q_{k-1}x_{i_{k-1}} + q_k x_{i_k})$$

$$y_2 := \frac{1}{Q_k}(q_k x_{i_1} + q_1 x_{i_2} + \ldots + q_{k-2}x_{i_{k-1}} + q_{k-1} x_{i_k})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{k-1} := \frac{1}{Q_k}(q_3 x_{i_1} + q_4 x_{i_2} + \ldots + q_1 x_{i_{k-1}} + q_2 x_{i_k})$$

$$y_k := \frac{1}{Q_k}(q_2 x_{i_1} + q_3 x_{i_2} + \ldots + q_k x_{i_{k-1}} + q_1 x_{i_k})$$

where $x_{i_j} \in \{x_1, \ldots, x_n\}$ and $q_j \geq 0$ $(j = 1, \cdots, k)$ are as above.

A simple computation shows that:

$$\frac{y_1 + y_2 + \ldots + y_{k-1} + y_k}{k} = \frac{x_{i_1} + x_{i_2} + \ldots + x_{i_{k-1}} + x_{i_k}}{k}$$

and the Jensen's inequality

$$\frac{1}{k}(f(y_1) + f(y_2) + \ldots + f(y_k)) \geq f\left(\frac{y_1 + \ldots + y_k}{k}\right)$$

yields that

$$\frac{1}{k}\{f[\frac{1}{Q_k}(q_1x_{i_1}+q_2x_{i_2}+\ldots+q_{k-1}x_{i_{k-1}}+q_kx_{i_k})]+\ldots$$
$$+f[\frac{1}{Q_k}(q_2x_{i_1}+q_3x_{i_2}+\ldots+q_kx_{i_{k-1}}+q_1x_{i_k})]\}$$
$$\geq f(\frac{x_{i_1}+\ldots+x_{i_k}}{k})$$

for all $i_1,\ldots,i_k=1,\cdots,n$.

Now, if we multiply this inequality with $p_{i_1}\ldots p_{i_k}\geq 0$ and if we sum after $i_1,\ldots,i_k$ to 1 at n, we derive:

$$\frac{1}{k}\{\sum_{i_1,\ldots,i_k=1}^{n}p_{i_1}\ldots p_{i_k}f[\frac{1}{Q_k}(q_1x_{i_1}+\ldots+q_kx_{i_k})]+\ldots$$
$$+\sum_{i_1,\ldots,i_k=1}^{n}p_{i_1}\ldots p_{i_k}f[\frac{1}{Q_k}(q_2x_{i_1}+\ldots+q_1x_{i_k})]\}$$
$$\geq\sum_{i_1,\ldots,i_k=1}^{n}p_{i_1}\ldots p_{i_k}f(\frac{x_{i_1}+\ldots+x_{i_k}}{k}).$$

Since the left membership of the above inequality is equal to

$$\sum_{i_1,\ldots,i_k=1}^{n}p_{i_1}\ldots p_{i_k}f[\frac{1}{Q_k}(q_1x_{i_1}+\ldots+q_kx_{i_k})]$$

the proof of the theorem is finished.

Corollary 1. *Let $f:C\subseteq X\to\mathbf{R}_+$ be a convex function such that f is also logarithmically concave on C, i.e., the mapping* $\log f$ *is concave on C. Then for all x_i, p_i $(i=1,\cdots,n)$, q_j $(j=1,\cdots,k)$ $(1\leq k\leq n)$ as above, one has the following refinement of the arithmetic mean-geometric mean inequality:*

$$\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)\geq\frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n}p_{i_1}\ldots p_{i_k}f(\frac{q_1x_{i_1}+\ldots+q_kx_{i_k}}{Q_k})$$

$$\geq \frac{1}{P_n^k} \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f(\frac{x_{i_1} + \dots + x_{i_k}}{k})$$
$$\geq f(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i)$$
$$\geq [\prod_{i_1,\dots,i_k=1}^{n} f^{p_{i_1} \cdots p_{i_k}}(\frac{x_{i_1} + \dots + x_{i_k}}{k})]^{1/P_n^k}$$
$$\geq [\prod_{i_1,\dots,i_k=1}^{n} f^{p_{i_1} \cdots p_{i_k}}(\frac{q_1 x_{i_1} + \dots + q_k x_{i_k}}{Q_k})]^{1/P_n^k}$$
$$\geq [\prod_{i=1}^{n} f^{p_i}(x_i)]^{1/P_n}.$$

Proof. The second part of Corollary 1 follows by the above theorem for the convex mapping - $\log f$. We will omit the details.

Now, let X be a real linear space and K a clin in X, i.e., a subset of X satisfying the conditions:

(K1) $x, y \in K$ imply $x + y \in K$;

(K2) $x \in K$, $\alpha \geq 0$ imply $\alpha x \in K$.

Also, let suppose that $\varphi : K \to \mathbb{R}$ is a quasi-linear functional on K, i.e., a mapping which verifies the assumption:

$$\varphi(\alpha x + \beta y) \leq (\geq) \alpha\varphi(x) + \beta\varphi(y),$$

for all $\alpha, \beta \geq 0$ and $x, y \in K$. We observe that such a functional is a convex (concave) mapping on K but the converse implication is not true, generally. We also observe that the following inequality holds (by induction):

$$\varphi(\sum_{i=1}^{n} p_i x_i) \leq (\geq) \sum_{i=1}^{n} p_i \varphi(x_i),$$

for all $p_i \geq 0$ and $x_i \in K$ $(i = 1, \dots, n)$.

By the use of the above theorem, we can improve this inequality as follows:

Corollary 2. *Let φ be a quasi-linear functional on K, $x_i \in K$ and $p_i \geq 0$ $(i = 1, \cdots, n)$ with $P_n > 0$. Then for all $q_j \geq 0$ $(j = 1, \cdots, k)$ $(1 \leq k \leq n)$ with*

$Q_k > 0$, *we have the inequalities*

$$\begin{aligned}\sum_{i=1}^{n} p_i\varphi(x_i) &\geq (\leq)\frac{1}{Q_k P_n^{k-1}} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}\varphi(q_1x_{i_1}+\ldots+q_kx_{i_k})\\ &\geq (\leq)\frac{1}{k\,P_n^{k-1}} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}\varphi(x_{i_1}+\ldots+x_{i_k})\\ &\geq (\leq)\varphi(\sum_{i=1}^{n} p_ix_i).\end{aligned}$$

Proof. The argument is obvious from the above theorem applied for the mapping φ.

3. Applications

1. Let $(X,\|\cdot\|)$ be a real normed linear space, $x_i \in X$ and p_i, q_j are as above. Then for all $p \geq 1$ we have the following refinement of the generalized triangle inequality:

$$\begin{aligned}P_n^{p-1}\sum_{i=1}^{n} p_i\|x_i\|^p &\geq \frac{P_n^{p-k}}{Q_k^p} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}\|q_1x_{i_1}+\ldots+q_kx_{i_k}\|^p\\ &\geq \frac{P_n^{p-k}}{k^p} \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k}\|x_{i_1}+\ldots+x_{i_k}\|^p\\ &\geq \|\sum_{i=1}^{n} p_ix_i\|^p.\end{aligned}$$

The proof follows by the above theorem for the convex mapping $f : X \to \mathbf{R}$, $f(x) := \|x\|^p$ $(p \geq 1)$.

2. Let $x_i > 0$, $p_i \geq 0$ $(i = 1,\ldots,n)$ and $q_j \geq 0$ $(j = 1,\cdots,k; 1 \leq k \leq n)$. Then the following refinement of weighted arithmetic mean-geometric mean inequality holds:

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_ix_i &\geq [\prod_{i_1,\ldots,i_k=1}^{n} (\frac{x_{i_1}+\ldots+x_{i_k}}{k})^{p_{i_1}\cdots p_{i_k}}]^{1/P_n^k}\\ &\geq [\prod_{i_1,\ldots,i_k=1}^{n} (\frac{q_1x_{i_1}+\ldots+q_kx_{i_k}}{Q_k})^{p_{i_1}\cdots p_{i_k}}]^{1/P_n^k}\\ &\geq (\prod_{i=1}^{n} x_i^{p_i})^{1/P_n}.\end{aligned}$$

The proof follows by Corollary 1 for the mapping $f : (0,\infty) \to (0,\infty)$, $f(x) := x$.

3. Now, let consider the mapping $f : (0, 1/2] \to (0, \infty)$ given by $f(x) := [x/(1-x)]^r$, $r \geq 1$. It is easily to see that f is convex on $(0, 1/2]$ and also logarithmically concave on this interval. By the use of Corollary 1, one has

$$\begin{aligned}\frac{1}{P_n}\sum_{i=1}^{n} p_i\left(\frac{x_i}{1-x_i}\right)^r \geq & \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\frac{q_1 x_{i_1}+\ldots+q_k x_{i_k}}{q_1(1-x_{i_1})+\ldots+q_k(1-x_{i_k})}\right)^r \\ \geq & \frac{1}{P_n^k}\sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\frac{x_{i_1}+\ldots+x_{i_k}}{k-x_{i_1}-\ldots-x_{i_k}}\right)^r \\ \geq & \left[\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i(1-x_i)}\right]^r \\ \geq & \left[\prod_{i_1,\ldots,i_k=1}^{n}\left(\frac{x_{i_1}+\ldots+x_{i_k}}{k-x_{i_1}-\ldots-x_{i_k}}\right)^{rp_{i_1}\cdots p_{i_k}}\right]^{1/P_n^k} \\ \geq & \left[\prod_{i_1,\ldots,i_k=1}^{n}\left(\frac{q_1 x_{i_1}+\ldots+q_k x_{i_k}}{q_1(1-x_{i_1})+\ldots+q_k(1-x_{i_k})}\right)^{rp_{i_1}\cdots p_{i_k}}\right]^{1/P_n^k} \\ \geq & \left[\prod_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)^{rp_i}\right]^{1/P_n}.\end{aligned}$$

Remark. The above inequalities contain refinements of C.-L. Wang's inequality [11] and also (if $p_i = 1$, $i = 1, \cdots, n$) of the well-known result of Ky Fan (see e.g. [3]):

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)} \geq \prod_{i=1}^{n}[x_i/(1-x_i)]^{1/n}.$$

4. As in [12], we shall use the following notations:

$\mathcal{M} =$ $\{M | M$ is a positive definite matrix of order $n\}$;

$|M| =$ the determinant of the matrix M;

$|M|_k = \prod_{i=1}^k \lambda_j$, $k = 1, \cdots, n$; where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M with $\lambda_1 \leq \ldots \leq \lambda_n$, $|M|_n = |M|$;

$M(j) =$ the submatrix of M obtained by deleting the j^{th} row and column of M;

$M[k] =$ the principal submatrix of M formed by taking the first k rows and columns of M, $M[n] = M$, $M[n-1] = M(n)$, $M[0] =$ the identity matrix;

BBF= the class of Bellman-Bergström-Fan quasi-linear functionals σ_i, δ_j and υ_k defined on $\mathcal{M}$ by

$$\sigma_i(M) := |M_i|^{1/i},\ i = 1, \ldots, n;$$
$$\delta_j(M) := |M|/|M(j)|,\ j = 1, \cdots, n;$$

and

$$\upsilon_k(M) := (|M|/|M[k]|)^{1/(n-k)},\ k = 1, \cdots, n;$$

respectively.

It is evident that $\mathcal{M}$ is closed under addition and multiplication by a positive number, i.e., $\mathcal{M}$ is a clin. Now, quasi-linearity of BBF-functionals follows from results in [12]:

$$\varphi(pM_1 + qM_2) \geq p\varphi(M_1) + q\varphi(M_2),$$

for all M_1, $M_2 \in \mathcal{M}$, $p, q \geq 0$ and $\varphi \in$BBF (see also [3] and [8]).

In [12], C.–L. Wang has obtained the following inequality:

$$\varphi(\sum_{i=1}^{m} p_i M_i) \geq \sum_{i=1}^{m} p_i \varphi(M_i) \geq P_m [\prod_{i=1}^{m} \varphi(M_i)]^{p_i/P_m} \tag{5}$$

where $p_i \geq 0$ $(i = 1, \ldots, m)$, which is an interpolating inequality for

$$\varphi(\frac{1}{P_m}\sum_{i=1}^{m} p_i M_i) \geq \prod_{i=1}^{m} [\varphi(M_i)]^{p_i/P_m}.$$

Note that (5) is also a generalization of a result from [9].

By the use of Corollary 2, we can improve the inequality (5) as follows:

$$\begin{aligned}
&\sum_{i=1}^{m} p_i \varphi(M_i) \leq \frac{1}{Q_k P_m^{k-1}} \sum_{i_1,\ldots,i_k=1}^{m} p_{i_1} \ldots p_{i_k} \varphi(q_1 M_{i_1} + \ldots + q_k M_{i_k}) \\
\leq& \frac{1}{k P_m^{k-1}} \sum_{i_1,\ldots,i_k=1}^{m} p_{i_1} \ldots p_{i_k} \varphi(M_{i_1} + \ldots + M_{i_k}) \\
\leq& \varphi(\sum_{i=1}^{m} p_i M_i)
\end{aligned}$$

where $1 \leq k \leq m$, $M_i \in \mathcal{M}$, $\varphi \in$BBF, $p_i \geq 0$ $(i = 1, \ldots, m)$ with $P_m > 0$ and $q_j \geq 0$ $(j = 1, \cdots, k; 1 \leq k \leq m)$ with $Q_k > 0$.

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