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JACOBI'S TWO-SQUARE AND FOUR-SQUARE THEOREMS VIA ROGER'S IDENTITY

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Abstract. We obtain Jacobi's two-square and four-square theorems as an application of an identity of L.J. Rogers.

1. Introduction

The problem of representation of an integer N as a sum of k squares is one of the most interesting problems in the theory of numbers and is treated in depth in the book E. Grosswald [4]. A great deal of attention has been paid to this problem in view of the fact that these representations have applications in many areas such as mechanics, crystallography and certain lattice point problems. Let $r_2(N)$ and $r_4(N)$ denote the number of representations of the positive integer N as a sum of two and four squares respectively and let $d_i(N)$ denote the number of divisors d of N, $d \equiv i \pmod 4$. Then, we have the well-know results of Jacobi:

Theorem 1.

$$r_2(N) = 4\left[d_1(N) - d_3(N)\right]. \tag{1}$$

Theorem 2.

$$r_4(N) = 8 \sum_{d|N, 4\nmid d} d. \tag{2}$$

Theorems on sum of squares are often treated as consequences of identities involving elliptic functions. Bilateral basic hypergeometric series have played an important role in the theory of representations of the numbers as sums of squares. For instance (1) and (2) have been proved in S. Bhargava and Chandrashekar Adiga [3] as a consequence of Ramanujan's ${}_1\Psi_1$-summation formula [8], [1]. Simple proofs of (1) and (2) have been given in M. D. Hirschhorn [5], [6] using Jacobi's triple product identity. G.E. Andrews

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[2] has proved these theorems as an application of W.N. Bailey's summation of the well-poised ${}_6\Psi_6$. The purpose of this note is to show that (1) and (2) can be easily deduced from the following identity of L.J. Rogers [9]:
If $|q| < 1$, $|abcd| < 1$, then

$$\sum_{n=0}^{\infty} \frac{(a/q)_n(1/b)_n(1/c)_n(1/d)_n(1-aq^{2n-1})(abcd)^n}{(ab)_n(ac)_n(ad)_n(q)_n(1-a/q)}$$
$$= \frac{(a)_\infty(abc)_\infty(abd)_\infty(acd)_\infty}{(ad)_\infty(ac)_\infty(ad)_\infty(abcd)_\infty} \tag{3}$$

where

$$\begin{aligned}(a)_0 &\equiv (a;q)_0 = 1,\\ (a)_n &\equiv (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}), \quad n \geq 1,\end{aligned}$$

and

$$(a)_\infty \equiv (a;q)_\infty = \lim_{m\to\infty} (a;q)_{m'} \ |q| < 1.$$

The above identity appears also as Entry 5, Chapter 16 of Ramanujan's second notebook [8]. It may be noted that this is a limiting case of a more general identity due to F.H. Jackson [7]. A proof of (3) may be found in C. Adiga et al. [1].

2. Proof of Theorem 1

Setting $b = c = -1$ and letting $d \to 0$, $a \to q$, we observe that the right hand side of (3) reduces to $(q)^2_\infty/(-q)^2_\infty$. Since,

$$\lim_{a\to q} \frac{(a/q)_n}{(1-a/q)} = (q)_{n-1}$$

and

$$\lim_{d\to 0}(1/d)_n d^n = (-1)^n q^{n(n-1)/2}$$

the left side of (3) reduces to

$$1 + \sum_{n=1}^{\infty} \frac{(q)_{n-1}(-1)_n(-1)_n(-1)^n q^{n(n-1)/2}(1-q^{2n})q^n}{(-q)_n(-q)_n(q)_n}$$

which is equal to

$$1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}.$$

Thus we have

$$\frac{(q)^2_\infty}{(-q)^2_\infty} = 1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}. \tag{4}$$

Putting $b=-1$ in (3) and letting $c\to 0$, $d\to 0$ $a\to q$ we get after some simplification of the left side

$$1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}=\frac{(q)_\infty}{(-q)_\infty}. \tag{5}$$

Using (5) in (4) and changing q to $-q$ we have

$$\begin{aligned}\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 =&1+4\sum_{n=1}^{\infty}\frac{(-1)^{n(n-1)/2}q^{n(n+1)/2}}{1+(-q)^n}\\ =&1+4\sum_{n=1}^{\infty}(-1)^{n(n-1)/2}q^{n(n+1)/2}\sum_{m=0}^{\infty}(-1)^m(-q)^{nm}\\ =&1+4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+n}q^{(2m+2n+1)n}\\ &+4\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)(n+1)}\sum_{m=0}^{\infty}q^{m(2n+1)}\\ =&1+4\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}(-1)^n q^{(2n+1)(m+1)}\\ &+4\sum_{n=0}^{\infty}(-1)^n q^{2n+1}\sum_{m=0}^{\infty}q^{(2n+1)(m+n)}\\ =&1+4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^n q^{(2n+1)(m+1)}\\ =&1+4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{(4n+1)(m+1)}-4\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{(4n+3)(m+1)}.\end{aligned}$$

Now, (1) follows upon comparing the coefficients of q^N on both sides of the above identity.

3. Proof of Theorem 2

Setting $b=c=d=-1$ in (3) and letting $a\to q$ we get

$$\begin{aligned}\frac{(q)_\infty^4}{(-q)_\infty^4}=&1+8\sum_{n=1}^{\infty}\frac{(-q)^n}{(1+q^n)^2}\\ =&1+8\sum_{n=1}^{\infty}\frac{(-1)^n n\, q^n}{1+q^n}\end{aligned} \tag{6}$$

using $x/(1+x)^2=-\sum_{m=1}^{\infty}m(-1)^m x^m$. Substituting (5) in (6) and changing q to $-q$ we get

$$\begin{aligned}\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^4 &= 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n} \\ &= 1 + 8\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + 8\sum_{n=1}^{\infty} \frac{2nq^{2n}}{1+q^{2n}} \\ &= 1 + 8\sum_{n=1}^{\infty} \left[\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}} + \frac{2nq^{2n}}{1-q^{2n}}\right] - 8\sum_{n=1}^{\infty} \left[\frac{2nq^{2n}}{1-q^{2n}} - \frac{2nq^{2n}}{1+q^{2n}}\right] \\ &= 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 8\sum_{n=1}^{\infty} \frac{4nq^{4n}}{1-q^{4n}} \\ &= 1 + 8\sum_{\substack{n=1 \\ 4\nmid n}}^{\infty} \frac{nq^n}{1-q^n}.\end{aligned}$$

Now, Theorem 2 follows directly by comparing the coefficients of q^N on both sides of the above identity.

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