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A GENERALIZATION OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. There are many classes of analytic functions in the unit disc U. We consider about the special classes $S_\lambda^*(A,B,\alpha,\beta)$ and $C_\lambda^*(A,B,\alpha,\beta)(-1 \le A < B \le 1,\ 0 < B \le 1,\ 0 \le \alpha < 1$ and $0 < \beta \le 1)$ of analytic functions in the unit disc U. And the purpose of this paper is to study the classes $S_\lambda^*(A,B,\alpha,\beta)$ and $C_\lambda^*(A,B,\alpha,\beta)$. We prove some distortion theorems and some coefficient estimates for these classes $S_\lambda^*(A,B,\alpha,\beta)$ and $C_\lambda^*(A,B,\alpha,\beta)$.

1. Introduction

Let $S_\lambda(\alpha,\beta)$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ and satisfy the condition

$$\left| \frac{\frac{f(z)}{g(z)} - 1}{\lambda \frac{f(z)}{g(z)} + 1} \right| < \beta \ (0 \le \lambda \le 1,\ 0 < \beta \le 1,\ z \in U),$$

where

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is analytic and starlike of order $\alpha (0 \le \alpha < 1)$ in the unit disc U, that is,

$$\text{Re}\left\{ \frac{zg'(z)}{g(z)} \right\} > \alpha$$

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for some $\alpha (0 \le \alpha < 1)$ and all $z \in U$. For this class $S_\lambda(\alpha, \beta)$, R.M. Goel and N.S. Sohi [3] showed a distortion theorem, the coefficient estimates and so on. In particular, the class $S_0(\alpha, \beta)$ was studied by R.M. Goel [2].

Let T denote the class of functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n (a_n \ge 0)$ which are analytic in the unit disc U. We use Ω to denote the class of analytic functions $w(z)$ in U satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For A, B fixed, $-1 \le A < B \le 1$, $0 < \beta \le 1$ and $0 \le \alpha < 1$, we say that a function $f(z)$ of T belongs to the class $T^*(A, B, \alpha)$ if and only if

$$\frac{zf'(z)}{f(z)} < \frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}, \quad z \in U \tag{1.1}$$

or equivalently, $f(z) \in T^*(A, B, \alpha)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}, \quad z \in U. \tag{1.2}$$

It follows from (1.2) that $f(z) \in T^*(A, B, \alpha)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{B\frac{zf'(z)}{f(z)} - [B + (A - B)(1 - \alpha)]} \right| < 1, \quad z \in U. \tag{1.3}$$

Further, $f(z)$ of T is said to belong to the class $C(A, B, \alpha)$ if and only if $zf'(z) \in T^*(A, B, \alpha)$. For these classes $T^*(A, B, \alpha)$ and $C(A, B, \alpha)$ Aouf [1] gave the following lemmas.

Lemma 1. *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \ge 0)$$

is in the class $T^*(A, B, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} [(1 + B)(n - 1) + (B - A)(1 - \alpha)] a_n \le (B - A)(1 - \alpha).$$

Lemma 2. *A function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \ge 0)$$

is in the class $C(A, B, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n[(1 + B)(n - 1) + (B - A)(1 - \alpha)] a_n \le (B - A)(1 - \alpha).$$

Lemma 3. *Let a function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

be in the class $T^*(A, B, \alpha)$. *Then*

$$|z| - \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}|z|^2 \leq |f(z)| \leq |z| + \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)}|z|^2$$

and

$$1 - \frac{2(B-A)(1-\alpha)}{[1+(B+(B-A)(1-\alpha)]}|z| \leq |f'(z)| \leq 1 + \frac{2(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}|z|.$$

Lemma 4. *Let a function*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

be in the class $C(A, B, \alpha)$. *Then*

$$|z| - \frac{(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)]}|z|^2$$
$$\leq |f(z)| \leq |z| + \frac{(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)]}|z|^2$$

and

$$1 - \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}|z| \leq |f'(z)| \leq 1 + \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}|z|.$$

Let $S^*_\lambda(A, B, \alpha, \beta)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disc U and satisfy the condition

$$\left| \frac{\frac{f(z)}{g(z)} - 1}{\lambda \frac{f(z)}{g(z)} + 1} \right| < \beta \quad (0 \leq \lambda \leq 1,\ 0 < \beta \leq 1,\ z \in U) \tag{1.4}$$

where

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0)$$

is in the class $T^*(A, B, \alpha)(-1 \leq A < B \leq 1,\ 0 < B \leq 1$ and $0 \leq \alpha < 1)$. And let $C^*_\lambda(A, B, \alpha, \beta)$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disc U and satisfy the condition (1.4) for

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0)$$

in the class $C(A, B, \alpha)(-1 \leq A < B \leq 1, 0 < B \leq 1$ and $0 \leq \alpha < 1)$.

2. Distortion theorems for the classes $S^*_\lambda(A, B, \alpha, \beta)$ and $C^*_\lambda(A, B, \alpha, \beta)$

Theorem 1. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $S^*_\lambda(A, B, \alpha, \beta)$. Then we have

$$|f(z)| \geq \frac{(1 - \beta|z|)\{[1 + B + (B - A)(1 - \alpha)] - (B - A)(1 - \alpha)|z|\}|z|}{[1 + B + (B - A)(1 - \alpha)](1 + \lambda\beta|z|)}$$

and

$$|f(z)| \leq \frac{(1 + \beta|z|)\{[1 + B + (B - A)(1 - \alpha)] + (B - A)(1 - \alpha)|z|\}|z|}{[1 + B + (B - A)(1 - \alpha)](1 - \lambda\beta|z|)}$$

for $z \in U$.

Proof. We employ the same technique as used by R.M. Goel and N.S. Sohi [3]. Since $f(z) \in S^*_\lambda(A, B, \alpha, \beta)$, after a simple computation we have

$$\frac{f(z)}{g(z)} = \frac{1 - \beta w(z)}{1 + \lambda\beta(z)}, \qquad w(z) \in \Omega.$$

Furthermore, by using Schwarz's Lemma [4], we have $|w(z)| \leq |z|$. Hence

$$\frac{1 - \beta|z|}{1 + \lambda\beta|z|} \leq \frac{|f(z)|}{|g(z)|} \leq \frac{1 + \beta|z|}{1 - \lambda\beta|z|}.$$

Consequently we have the theorem with the aid of Lemma 3.

Corollary 1. *Under the hypotheses of Theorem 1, $f(z)$ is included in the disc with center at the origin and radius*

$$\frac{(1+\beta)\{1+B+2(B-A)(1-\alpha)\}}{[1+B+(B-A)(1-\alpha)](1-\lambda\beta)}.$$

Theorem 2. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $C_{\lambda}^{*}(A,B,\alpha,\beta)$. Then we have

$$|f(z)|\geq\frac{(1-\beta|z|)\{2[1+B+(B-A)(1-\alpha)]-(B-A)(1-\alpha)|z|\}|z|}{2[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)}$$

and

$$|f(z)|\leq\frac{(1+\beta|z|)\{2[1+B+(B-A)(1-\alpha)]+(B-A)(1-\alpha)|z|\}|z|}{2[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)}$$

for $z\in U$.

The proof of Theorem 2 is obtained by using the same techique as in the proof of Theorem 1 with the aid of Lemma 4.

Corollary 2. *Under the hypotheses of Theorem 2, $f(z)$ is included in the disc with center at the origin and radius*

$$\frac{(1+\beta)\{2(1+B)+3(B-A)(1-\alpha)\}}{2[1+B+(B-A)(1-\alpha)](1-\lambda\beta)}.$$

Theorem 3. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $S_{\lambda}^{*}(A,B,\alpha,\beta)$. Then we have

$$|f'(z)|\leq\frac{(1+\beta|z|)\{[1+B+(B-A)(1-\alpha)]+2(B-A)(1-\alpha)|z|\}}{[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)}+\frac{(1+\lambda)\beta\{[1+B+(B-A)(1-\alpha)]+(B-A)(1-\alpha)|z|\}|z|}{[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)^2(1-|z|^2)}$$

for $z\in U$.

Proof. Since $f(z) \in S_\lambda^*(A, B, \alpha, \beta)$, by using

$$\frac{f(z)}{g(z)} = \frac{1 - \beta w(z)}{1 + \lambda\beta w(z)}, \qquad w \in \Omega,$$

we obtain

$$f'(z) = \frac{1 - \beta w(z)}{1 + \lambda\beta w(z)} g'(z) - \frac{(1+\lambda)\beta w'(z)}{\{1 + \lambda\beta w(z)\}^2} g(z).$$

Moreover we have $|w'(z)| \le \frac{1}{(1-|z|^2)}$ by means of Caratheodory's theorem [4]. Hence we obtain the theorem with the aid of Lemma 3.

Theorem 4. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $C_\lambda^*(A, B, \alpha, \beta)$, Then we have

$$\begin{aligned} |f'(z)| \le & \frac{(1+\beta|z|)\{[1+B+(B-A)(1-\alpha)] + (B-A)(1-\alpha)|z|\}}{[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)} \\ & + \frac{(1+\lambda)\beta\{2[1+B+(B-A)(1-\alpha)] + (B-A)(1-\alpha)|z|\}|z|}{2[1+B+(B-A)(1-\alpha)](1-\lambda\beta|z|)^2(1-|z|^2)} \end{aligned}$$

for $z \in U$.

The proof of Theorem 4 is obtained by using the same technique as in the proof of Theorem 3 with the aid of Lemma 4.

3. The coefficient estimates

Theorem 5. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $S_\lambda^*(A, B, \alpha, \beta)$. Then we have

$$|a_2| \le \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]} + \beta(1+\lambda)$$

and

$$\begin{aligned} |a_3| \le & \frac{(B-A)(1-\alpha)}{[2(1+B)+(B-A)(1-\alpha)]} \\ & + \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}\beta(1+\lambda) + \lambda\beta^2(1+\lambda). \end{aligned}$$

Proof. Let

$$w(z) = \sum_{n=2}^{\infty} c_n z^n$$

be analytic in the unit disc U and satisfy the condition $|w(z)| < 1$. Then we obtain $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$ [4]. Since $f(z) \in S_\lambda^*(A, B, \alpha, \beta)$, by using

$$\frac{f(z)}{g(z)} = \frac{1 - \beta w(z)}{1 + \lambda\beta w(z)},$$

we have

$$(z + \sum_{n=2}^{\infty} a_n z^n)(1 + \lambda\beta \sum_{n=1}^{\infty} c_n z^n) = (z - \sum_{n=2}^{\infty} b_n z^n)(1 - \beta \sum_{n=1}^{\infty} c_n z^n). \tag{3.1}$$

Equating coefficients of z^2 and z^3 on both sides of (3.1), we obtain

$$a_2 = -\beta(1+\lambda)c_1 - b_2$$

and

$$a_3 = -\beta(1+\lambda)c_2 + \lambda\beta^2(1+\lambda)c_1^2 + \beta(1+\lambda)b_2 c_1 - b_3.$$

Since $g(z) \in T^*(A, B, \alpha)$, by using Lemma 1,

$$b_2 \leq \frac{(B-A)(1-\alpha)}{[1 + B + (B-A)(1-\alpha)]}$$

and

$$b_3 \leq \frac{(B-A)(1-\alpha)}{[2(1+B) + (B-A)(1-\alpha)]}.$$

Hence we have th : theorem.

Theorem 6. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $C_\lambda^*(A, B, \alpha, \beta)$. *Then we have*

$$|a_2| \leq \frac{(B-A)(1-\alpha)}{2[1 + B + (B-A)(1-\alpha)]} + \beta(1+\lambda)$$

and

$$|a_3| \leq \frac{(B-A)(1-\alpha)}{3[2(1+B) + (B-A)(1-\alpha)]} + \frac{(B-A)(1-\alpha)}{2[1 + B + (B-A)(1-\alpha)]}\beta(1+\lambda) + \lambda\beta^2(1+\lambda).$$

The proof of Theorem 6 is given in much the same way as Theorem 5 with the sid of Lemma 2.

Remark. Putting $A=-1$ and $B=1$ in the above results, we get the results of S. Owa [5].

Theorem 7. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

be in the class $S_\lambda^*(A,B,\alpha,\beta)$. *Then we have*

$$\begin{aligned}|a_4|\le&\frac{(B-A)(1-\alpha)}{[3(1+B)+(B-A)(1-\alpha)]}+\frac{(B-A)(1-\alpha)}{[2(1+B)+(B-A)(1-\alpha)]}\beta(1+\lambda)\\&+\frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}\beta(1+\lambda)(1+\lambda\beta)+\beta(1+\lambda)\\&+\lambda\beta^2(1+\lambda)+\lambda^2\beta^3(1+\lambda).\end{aligned}$$

Proof. Equating the coefficients of z^4 on both sides of (3.1), we have

$$a_4=-b_4-\beta(1+\lambda)c_3-\beta(\lambda a_2-b_2)c_2-\beta(\lambda a_3-b_3)c_1.$$

Since

$$c_n=\frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)dz}{z^{n+1}}\qquad(0<r<1,\ n\in N)\tag{3.2}$$

for the coefficients c_n of analytic function $w(z)$ in the unit disc U,

$$|a_4|\le|b_4|+\frac{\beta(1+\lambda)}{r^3}+\frac{\beta(\lambda|a_2|+|b_2|)}{r^2}+\frac{\beta(\lambda|a_3|+|b_3|)}{r}.$$

Furthermore, since the above inequality holds for any $r(0<r<1)$, we obtain

$$|a_4|\le|b_4|+\beta(1+\lambda)+\beta(\lambda|a_2|+|b_2|)+\beta(\lambda|a_3|+|b_3|).$$

Hence we obtain the theorem by using Lemma 1 and Theorem 5.

Theorem 8. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

be in the class $C^*_\lambda(A,B,\alpha,\beta)$. *Then we have*

$$|a_4| \leq \frac{(B-A)(1-\alpha)}{4[3(1+B)+(B-A)(1-\alpha)]} + \frac{(B-A)(1-\alpha)}{3[2(1+B)+(B-A)(1-\alpha)]}\beta(1+\lambda)$$
$$+ \frac{(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)]}\beta(1+\lambda)(1+\lambda\beta) + \beta(1+\lambda)$$
$$+ \lambda\beta^2(1+\lambda) + \lambda^2\beta^3(1+\lambda).$$

The proof of Theorem 8 is obtained by using the same technique as in the proof of Theorem 7 with Lemma 2 and Theorem 6.

Theorem 9. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $S^*_0(A,B,\alpha,\beta)$. *Then we have*

$$|a_n| \leq \beta\frac{(1+B)+2(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]} + \frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]}$$

for any $n \geq 2$.

Proof. Since $f(z) \in S^*_0(A,B,\alpha,\beta)$, after a simple computation, we have

$$\frac{f(z)}{g(z)} = 1 - \beta w(z), \qquad w \in \Omega. \tag{3.3}$$

Let

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \qquad (z \in U).$$

Then, on substituting the power series for functions $f(z)$, $g(z)$ and $w(z)$ in (3.3), we obtain

$$-\frac{1}{\beta}\sum_{n=2}^{\infty}(a_n+b_n)z^n = (z - \sum_{n=2}^{\infty} b_n z^n)(\sum_{n=1}^{\infty} c_n z^n).$$

Equating the coefficients of z^n on both sides of the above equality, we have

$$-\frac{1}{\beta}(a_n+b_n) = c_{n-1} - \sum_{m=2}^{n-1} b_m c_{n-m}.$$

using (3.2), we obtain

$$-\frac{1}{\beta}(a_n+b_n) = \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)}{z^n}(1 - \sum_{m=2}^{n-1} b_m z^{m-1})dz.$$

Furthermore, by using Schwarz's Lemma [4], we have $|w(z)| \leq |z|$ for any $z \in U$. Consequently

$$\begin{aligned}\frac{1}{\beta}|a_n + b_n| \leq & \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{r^{n-1}}\left|1 - \sum_{m=2}^{n-1} b_m r^{m-1} e^{(m-1)i\theta}\right| d\theta \\ \leq & \frac{1}{2\pi r^{n-1}}\int_0^{2\pi} (1 + \sum_{m=2}^{n-1} b_m r^{m-1}) d\theta \\ = & \frac{1}{r^{n-1}}(1 + \sum_{m=2}^{n-1} b_m r^{m-1}) \\ \leq & \frac{1}{r^{n-1}}(1 + \sum_{m=2}^{n-1} b_m).\end{aligned}$$

Since the above inequality holds for any $r(0 < r < 1)$ and

$$\sum_{m=2}^{n-1} b_m \leq \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}$$

for any $n \geq 2$ by Lemma 1, we have

$$\frac{1}{\beta}|a_n + b_n| \leq 1 + \frac{(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]}.$$

Hence we obtain

$$\begin{aligned}|a_n| \leq & |a_n + b_n| + |b_n| \\ \leq & \beta\frac{(1+B)+2(B-A)(1-\alpha)}{[1+B+(B-A)(1-\alpha)]} + \frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]}\end{aligned}$$

because $b_n \leq \frac{(B-A)(1-\alpha)}{[(1+B)(n-1)+(B-A)(1-\alpha)]}$ for any $n \geq 2$ by Lemma 1.

Theorem 10. *Let a function*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $C_0^*(A, B, \alpha, \beta)$. Then we have

$$|a_n| \leq \beta\frac{2(1+B)+3(B-A)(1-\alpha)}{2[1+B+(B-A)(1-\alpha)]} + \frac{(B-A)(1-\alpha)}{n[(1+B)(n-1)+(B-A)(1-\alpha)]}$$

for any $n \geq 2$.

The proof of theorem 10 is obtained by using the same technique as in the proof of Theorem 9 with the aid of Lemma 2.

Remark. Putting $A = -1$ and $B = 1$ in Theorem 7, 8, 9 and 10 we get the results of S. Owa [6].

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