TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 1, Spring 1994

ON FILTER EXPANSION OF TOPOLOGIES

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Abstract. The lower separation axioms $T_i, (i \in \{0,1,2\})$ are obviously preserved under topology expansions. This fact is not generally valid for higher separation axioms as well as for recent sorts of separation such as T_R, R_0, R_1 and semi-R_i, $(i \in \{0,1\})$. The purpose of the present work is to investigate preservation of these recent separation properties under filter expansion of topologies. Also, we study the effect of filter expansions on the concept s-essentially T_i- spaces, $(i \in \{0,1\})$

1. Introduction

For any subset A of a topological space (X,τ) we denote the closure (resp. the interior) of A with respect to τ by $\text{cl}_\tau A$ (resp. $\text{int}_\tau A$). A subset A is said to be semiopen [151 (resp. α-open [17]) if $A \subset$ cl (int A) (resp. $A \subset$ int (cl (int A). A is semiclosed [4] if $(X - A)$ is semiopen. We denote the family of all semiopen (resp. semi-closed, α-open) sets in (X,τ) by $SO(X,\tau)$ (resp. $SC(x,\tau)$, τ^α)). Njasted [17] has shown that τ^α is a topology on X satisfying $\tau \subset \tau^\alpha$. The semi closure of A [4] (denoted by s-cl A) is the intersection of all semiclosed sets containing A. A space X is submaximal [3] iff all dense subsets are open and is resolvable [11], if there is a subset D of X such that D and $X - D$ are both dense in X and it is irresolvable if it is not resolvable.

Definition 1.1. [1] Let (X,τ) be a space and $\mathcal{F}$ be a filter on X, then the topology $\tau(\mathcal{F}) = \{U \cap F : U \in \tau, F \in \mathcal{F}\}$ is called a filter expansion of τ. We remark that $d_{\tau(\mathcal{F})}A \subset d_\tau A$, for any $A \subset X$, where $d_{\tau(\mathcal{F})}, d_\tau$ denote the derived operators relative to $\tau(\mathcal{F})$ and τ, respectively

Definition 1.2. [18] A space (X,τ) is T_R iff for every $x \in X$, either $d\{x\}$ is empty or $d\{x\}$ contains a non-empty closed set.

Received August 30, 1992.

1991 AMS Subject Classification : 54 A 10, 54 C10, 54 D 20.

Key words and phrases: Semiopen and α-open sets, filter expansion of topologies, semi-T_0-identification space, T_R, R_0, R_1 and semi-R_i, $i \in \{0,1\}$.

Definition 1.3. [18] A space (X,τ) is C_R iff for every $x \in X$, $d\{x\}$ does not contain a non-empty closed set.

Definition 1.4. [6] In a space (X,τ), let $x \in X$. Then the kernal of x, $\{\hat{x}\} = \cap\{O/x \in O, O \in \tau\}$ and the covering of x, $< x >= \text{cl}\{x\} \cap \{\hat{x}\}$

Definition 1.5. [13] In a space (X,τ), we will call superderived of x, $x \in X$, the set $D\{x\} = \text{cl}\{x\} - < x >$, that is the union of the closed sets which are contained in $d\{x\}$.

Definition 1.6. [5] A space (X,τ) is R_0 iff for orbitrary $x,y \in X$, either cl $\{x\} = \text{cl}\{y\}$ or $\text{cl}\{x\} \cap \text{cl}\{y\} = \emptyset$

Definition 1.7. [5] A space (X,τ) is R_1 iff for each pair $x,y \in X$ such that $\text{cl}\{x\} \neq \text{cl}\{y\}$, there exist disjoint open sets U and V such that $\text{cl}\{x\} \subset U$ and $\text{cl}\{y\} \subset V$. In [7] semi-R_0 and semi-R_1 are defined by the same manner by replacing the word open by semiopen and closure by semiclosure.

Definition 1.8. [13] *A* space (X,τ) is a R^*_{YS}- space if for arbitrary x, $y \in X$, $x \neq y$ implies $D\{x\} \cap D\{y\} = \emptyset$

Definition 1.9. [19] Let R be the equivalence relation on space (X,τ) defined by xRy iff $\text{cl}\{x\} = \text{cl}\{y\}$. Then the T_0-identification space of (X,τ) is $(X_0, Q(X_0))$ where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0.

Definition 1.10. [9] Let R be the equivalence relation on a space (X,τ) defined by xRy iff s-cl$\{x\}$ = s-cl$\{y\}$. Then the semi-T_0 -identification space of (X,τ) is $(X_s, Q(X_s))$, where X_s is the set of equivalence classes of R and $Q(X_s)$ is the decomposition topology on X_s. For each $x \in X$, let k_x be the equivalence class of R containing x.

Definition 1.11. [9] A space (X,τ) is s-essentially T_x iff $(X_s, Q(X_s))$ is T_x.

Definition 1.12. [14] A space (X,τ) is ET_x (essentially T_x) if its T_0-identification space is T_x.

2. Separation Properties

Lemma 2.1. [12] *A space (X,τ) is a T_R iff any non-empty closed subset of X contains a closed singleton set.*

Theorem 2.1. (a) *If (X,τ) is T_R, then $(X,\tau(\mathcal{F}))$ is T_R.*
(b) *If (X,τ) is C_R, then $(X,\tau(\mathcal{F}))$ is also C_R.*

Proof. (a) let $\emptyset \neq A$ be a closed set in $\tau(\mathcal{F})$, then $A = W \cup B$, where $W \in \tau^c$, $B \notin \mathcal{F}$. By lemma 2.1, we have that $\text{cl}_\tau\{x\} \subset W \subset A$. Thus $\text{cl}_{\tau(\mathcal{F})}\{x\} \subset A$ and

$(X, \tau(\mathcal{F}))$ is T_R.

(b) It is similar to the proof of (a)

Lemma 2.2. *If (X, τ) is a space, and $(X, \tau(\mathcal{F}))$ is a filter expansion of τ, then for each $x \in X$; the following are hold:*

(a) $\{\hat{x}\}_{\tau(\mathcal{F})} \subset \{\hat{x}\}_\tau$

(b) $< x >_{\tau(\mathcal{F})} \subset < x >_\tau$

(c) $D_{\tau(\mathcal{F})}\{x\} \subset D_\tau\{x\}$

Proof. (a) Since $\tau \subset \tau(\mathcal{F})$, then it is obvious.

(b) By (a), the proof is obvious.

(c) Since, $\text{cl}_{\tau(\mathcal{F})}\{x\} \subset \text{cl}_\tau\{x\}$, then $\text{cl}_{\tau(\mathcal{F})}\{x\} - < x >_\tau \subset \text{cl}_\tau\{x\} - < x >_\tau$.

Hence $\text{cl}_{\tau(\mathcal{F})}\{x\} - < x >_{\tau(\mathcal{F})} \subset \text{cl}_{\tau(\mathcal{F})}\{x\} - < x >_\tau \subset \text{cl}_\tau\{x\} - < x >_\tau$.

Thus $D_{\tau(\mathcal{F})}\{x\} \subset D_\tau\{x\}$.

Lemma 2.3 *[16] A space (X, τ) is R_0 iff one of the following conditions are hold:*

a) for each $x \in X$, $cl\{x\} = < x > = \{\hat{x}\}$

b) for each $x \in X$, $D\{x\} = \emptyset$.

Theorem 2.2 *If (X, τ) is R_0 (resp. R_1), then $(X, \tau(\mathcal{F}))$ is R_0 (resp. R_1).*

Proof. By lemmas 2.2 and 2.3.

Lemma 2.4. [1] *If (X, τ) is an irresolvable space and $\tau(\mathcal{F})$ is a filter expansion of τ by afilter $\mathcal{F}$ on X and $F \in SO(X, \tau)$, for every $F \in \mathcal{F}$, then $SO(X, \tau) = SO(X, \tau(\mathcal{F}))$.*

Lemma 2.5. *If (X, τ) is an irresolvable space and $\tau(\mathcal{F})$ is a filter expansion of τ by a filter $\mathcal{F}$ on X such that $F \in SO(X, \tau)$, for every $F \in \mathcal{F}$, then for each $A \subset X$, s-$cl_\tau A$ = s-$cl_{\tau(\mathcal{F})} A$.*

Proof. Since $A \subset$ s-$\text{cl}_{\tau(\mathcal{F})}A$, and s-$\text{cl}_{\tau(\mathcal{F})}A \in SC(X, \tau(\mathcal{F})) = SC(X, \tau)$ [By lemma 2.4]. Thus s-$\text{cl}_\tau A \subset$ s-$\text{cl}_{\tau(\mathcal{F})}A$. But s-$\text{cl}_{\tau(\mathcal{F})}A \subset$ s-$\text{cl}_\tau A$. Hence s-$\text{cl}_\tau A$ = s-$\text{cl}_{\tau(\mathcal{F})}A$.

Lemma 2.6. *[1] If (X, τ) is an irresolvable space and $\tau(\mathcal{F})$ is a filter expansion of τ by a filter $\mathcal{F}$ on x, then $SO(X, \tau) \subset SO(X, \tau(\mathcal{F}))$.*

Theorem 2.3. *If (X, τ) is an irresolvable and semi-R_i. Then $(X, \tau(\mathcal{F}))$ is semi-R_i, ($i \in \{0, 1\}$)*

Proof. We will give the proof for semi-R_1, Let $x, y \in X$ such that s-$\text{cl}_{\tau(\mathcal{F})}\{x\} \neq$ s-$\text{cl}_{\tau(\mathcal{F})}\{y\}$, implies that, s-$\text{cl}_\tau\{x\} \neq$ s-$\text{cl}_\tau\{y\}$. Since (X, τ) is semi- R_1, there exist $U, V \in SO(X, \tau)$ such that s-$\text{cl}_\tau\{x\} \subset U$ and s-$\text{cl}_\tau\{y\} \subset V$, and $U \cap V = \emptyset$. But

$SO(X,\tau) \subset SO(X,\tau(\mathcal{F}))$. Thus $(X,\tau(\mathcal{F}))$ is semi-R_1. The proof is similar for semi-R_0, spaces.

Theorem 2.4. *If (X,τ) is an irresolvable space, $\tau(\mathcal{F})$ is a filter expansion of τ by a filter $\mathcal{F}$ on X such that $F \in SO(X,\tau)$, for every $F \in \mathcal{F}$. Then (X,τ) is semi-R_i iff $(X,\tau(\mathcal{F}))$ is semi-R_1, $(i \in \{0,1\})$*

Proof. From theorem 2.3, if (X,τ) is semi-R_1, then $(X,\tau(\mathcal{F}))$ is also. To prove the converse, let $x,y \in X$ such that s-$\mathrm{cl}_\tau\{x\} \neq$ s-$\mathrm{cl}\{y\}$, implies that s-$\mathrm{cl}_{\tau(\mathcal{F})}\{x\} \neq$ s-$\mathrm{cl}_{\tau(\mathcal{F})}\{y\}$. Thus there exist $U,V \in SO(X,\tau(\mathcal{F}))$ such that s-$\mathrm{cl}_{\tau(\mathcal{F})}\{x\} \subset U$ and s-$\mathrm{cl}_{\tau(\mathcal{F})}\{y\} \subset V$ and $U \cup V = \phi$. By lemmas 2.4 and 2.5, we arrive that (X,τ) is semi-R_1. The proof is similar for semi-R_0 spaces.

Theorem 2.5. *If (X,τ) is R^*_{YS} then $(X,\tau(\mathcal{F}))$ is also R^*_{YS}.*

Proof. By lemma 2.2 (c)

Lemma 2.7. (a) *If A is nowhere dense in a space X, then $(X-A)$ is dense.*
(b) *If A is dense and open subset of a space X, then $(X-A)$ is nowhere dense.*

Proof. Obvious

Lemma 2.8. [17] *Let (X,τ) be a space and $A \subset X$. Then $A \in \tau^\alpha$ iff $A = U - N$, such that $U \in \tau$ and N is a nowhere dense set.*

Remark 2.1. If (X,τ) is a submaximal space, then all dense subsets is a filter on X.

Theorem 2.6. *If (X,τ) is a submaximal space and $\mathcal{F}$ is a filter on X such that all elements are dense in X. Then $\tau(\mathcal{F}) = \tau^\alpha$*

Proof. Let $W \in \tau(\mathcal{F})$, then $W = U \cap A$, $U \in \tau$ and $A \in \mathcal{F}$. Thus $W = U - A^c$, and by lemma 2.7, A^c is nowhere dense in X. By lemma 2.8, $W \in \tau^\alpha$ and $\tau(\mathcal{F}) \subset \tau^\alpha$. Conversely, let $W \in \tau^\alpha$, then $W = U - N$, $U \in \tau$ and N is nowhere dense in τ. Thus $N^c \in \mathcal{F}$ and $W \in \tau(\mathcal{F})$. Hence $\tau^\alpha \subset \tau(\mathcal{F})$.

Lemma 2.9. [2] *A subset A of a space (X,τ) is dense iff $A \cap U \neq \emptyset$ for each $U \in \tau - \{\emptyset\}$*

Remark 2.2. It is easy to prove that lemma 2.9 still valid on replacing the word open by semi-open

Theorem 2.7 *If (X,τ) is an irresolvable space and $\tau(\mathcal{F})$ is a filter expansion of τ by a filter $\mathcal{F}$ on X and $F \in SO(X,\tau)$, for every $F \in \mathcal{F}$. Then τ and $\tau(\mathcal{F})$ have identical dense sets.*

Proof. By Lemma 2.4 and Remark 2.2.

3. S-Essentially T_i-Spaces, $(i \in \{0,1\})$

Theorem 3.1 [9] *Let (X,τ) be a space. Then the following are equivalent:*
(a) *(X,τ) is s-essentially To*
(b) $X_s = X_0$
(c) *If $x \in X$ such that $C_x \neq \{x\}$ and $O \in \tau$ such that $x \in cl_\tau O$. Then $x \in O$,*
(d) *If $x \in X$ such that $C_x \neq \{x\}$ and $O \in SO(X,\tau)$ such that $x \in O$. Then $x \in int_\tau(O)$.*

Theorem 3.2. *If (X,τ) is s-essentially T_0, then $(X,\tau(\mathcal{F}))$ is also s-essentially To.*

Proof. Let $x \in X$ such that $C_x \neq \{x\}$. $O \in \tau(\mathcal{F})$ and $x \in \mathrm{cl}_{\tau(\mathcal{F})}O$. Then $O = U \cap F$, $U \in \tau$ and $F \in \mathcal{F}$ and $\mathrm{cl}_{\tau(\mathcal{F})}O = \mathrm{cl}_{\tau(\mathcal{F})}(U \cap F) \subset \mathrm{cl}_{\tau(\mathcal{F})}U \cap \mathrm{cl}_{\tau(\mathcal{F})}F \subset \mathrm{cl}_{\tau(\mathcal{F})}U \subset \mathrm{cl}_\tau U$. Hence $x \in \mathrm{cl}_\tau U$. But (X,τ) is s-essentially T_0, implies that $x \in U$. There are two cases:
(a) If $x \in F$, then $x \in U \cap F = O$. By Theorem 3.1 $(X,\tau(\mathcal{F}))$ is s-essentially To.
(b) If $x \notin F$, then $x \notin U \cap F = O$. Let $y \in C_x$ such that $y \neq x$ implies that $\mathrm{cl}_{\tau(\mathcal{F})}\{y\} = \mathrm{cl}_{\tau(\mathcal{F})}\{x\} \subset \mathrm{cl}_{\tau(\mathcal{F})}O$, since $x \notin O$, then $x \in (X-O) \in [\tau(\mathcal{F})]^c$. Hence $\mathrm{cl}_{\tau(\mathcal{F})}\{x\} \subset X-O$, which implies that $y \notin \mathrm{cl}_{\tau(\mathcal{F})}\{x\}$. This is a contractiction .

Theorem 3.3. [9] *If (X,τ) is a noncompact space and $p \notin X$ such that $X^* = X \cup \{p\}$. Let τ^* be the one-point compactification topology on X^*, and let i: $(X,\tau) \to (X^*,\tau^*)$ be the identity mapping. Then (X,τ) is s-essentially To iff (X^*,τ^*) is s-essentially T_0.*

Theorem 3.4 *If (X,τ) is a noncompact space and $p \notin X$ such that $X^* = X \cup \{p\}$, and τ^* is the one -point compactification topology on X^*, and let i: $(X,\tau) \to (X^*,\tau^*)$ be the identity mapping and (X,τ) is s-essentially T_0. Then $(X,\tau(\mathcal{F}))$ and $(X^*,\tau^*(\mathcal{F}))$ are s-essentially T_0.*

Proof. By theorems 3.2 and 3.3

Definition 3.1. [8] If f is a continuous function from a space (X,τ) onto (Y,S), then the function $f^* : (X_0, Q(\tau)) \to (Y_0, Q(S))$ defined by $f^*(C_x) = C_{f(x)}$ is the induced map from $(X_0, Q(\tau))$ onto $(Y_0, Q(S))$ determined by f.

Theorem 3.5. [8] *If f is a continuous function from (X,τ) onto (Y,S), then the relation $f^* = \{(C_x, C_{f(x)})/C_x \in X_0\}$ is a continuous function from $(X_0, Q(\tau))$ onto $(Y_0, Q(S))$.*

Theorem 3.6. [8] *Let f be a continuous function from (X,τ) onto (Y,S). If*

f is open (closed), then f^* is open (closed).

Theorem 3.7. *Let $f : (X, \tau) \to (X, \tau(\mathcal{F}))$ be a homeomorphic mapping and (X, τ) be a compact space, then:*
(i) $f^* : (X_0, Q(\tau)) \to (X_0, Q(\tau(\mathcal{F}))$ is also a homeomorphic mapping.
(ii) $(X, \tau(\mathcal{F}))$, $(X_0, Q(\tau))$, $(X_0, Q(\tau(\mathcal{F}))$ are also compact spaces.

Proof. By using theorems 3.5 and 3.6.

By Theorem 3.7, we give the following diagram (where P_q is projection mapping of X onto X_0 [19].

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (X,\tau(\mathcal{F})) \\ {\scriptstyle P_q}\downarrow & & {\scriptstyle P_q^*}\downarrow \\ (X_0,Q(\tau)) & \xrightarrow{f^*} & (X_0,Q(\tau(\mathcal{F}))) \end{array}$$

Theorem 3.8. [16] *A space (X, τ) is ET_{YS} iff for $x, y \in X, < x > \neq < y >$ implies $D\{x\} \cap D\{y\} = \emptyset$*

Theorem 3.9. *If (X, τ) is ET_{YS}, then $(X, \tau(\mathcal{F}))$ is also ET_{YS}.*

Proof. By Theorem 3.8, and Lemma. 2.2

Theorem 3.10. *If (X, τ) is ET_1 (or ET_D), then $(X, \tau(\mathcal{F}))$ is ET_1 (or ET_D).*

Proof. By Lemma 2.2., the proof is obvious.

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