TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 1, Spring 1994

EXISTENCE OF SOLUTIONS FOR ELLIPTIC INTEGRO-DIFFERENTIAL SYSTEMS

LONG-YI TSAI AND S.T. WU

Abstract. In this paper the existence of the solution for elliptic integro-differential systems are discussed. Those systems are motivated by certain physical processes such as in epidemics, predator-prey dynamics and the others. We extend the method of mixed monotony to second order elliptic partial integro-differential equations. By assuming the existence of a satellite f of the give function Φ, we prove the existence of solutions by using fixed point theory. Moreover, we provide the modified method of mixed monotony to construct two monotone sequences which converge uniformly to the solution. We also give sufficient conditions for the existence of f and obtain the construction of upper and lower solutions in some applications.

1. Introduction

The existence of the solutions for elliptic integro-differential systems is discussed. Those systems are motivated by many physical processes in epidemics, predator-prey dynamics and the others (see [4,6,9]). Recently, the method of mixed monotony [2,3] or the condition of heterotony [5] are employed to construct monotone sequences that converge to the solution of initial (boundary) value problem for first (second) order ordinary differential equations when the function Φ involved do not possess any monotone properties. In this paper, we shall extend this method to second order elliptic partial integro-differential equations. The content of this paper is organized as follows. In section 2, some notations and preliminary lemmas are given. In section 3, we first assume the existence of the satellite f of Φ and prove the existence of the solutions by using Schauder fixed point theorem. And then we give the sufficient condition for the existence of f. The construction of ϵ-upper and ϵ-lower solutions are obtained in some examples. In the last section, by assuming the existence of f which is ϵ-monotone nondecreasing in v, monotone nondecreasing in p and monotone nonincreasing in w and q, we find two

Received September 2, 1992.
1991 AMS Subject classification: 11E25, 11P05.
Keywords: Satellite, ϵ-upper(ϵ-lower) solutions, upper(lower) solutions..

monotone sequences which converge to the solution uniformly. Sufficient condition for the existence of f is also given. Analogous existence results are easily obtained when Φ does not depend on integral terms. For parabolic integro-differential systems, parallel results will be given in the forthcoming paper.

2. Definitions and Notation

Let R^N denote a N-dimensional real Euclidean space and let Ω be a bounded domain in R^N whose boundary $\partial\Omega$ is a $(N-1)$-dimensional manifold of class $C^{2+\lambda}$, $\lambda \in (0,1)$. Let $C(\overline{\Omega})$ be the Banach space of continuous functions with domain $\overline{\Omega}$ and range in R endowed with the usual maximun norm. Let $C^{\ell}(\overline{\Omega})$ be a Banach space, i.e., the set of $u \in C(\overline{\Omega})$ such that u belongs to the class $C^{[\ell]}$ with respect to x and the $[\ell]$-th partial derivatives of u with respect to the components of x are Hölder continuous with exponent $\ell - [\ell]$, here $[\ell]$ is the greatest integer $\leq \ell$. For $u = (u_1, \ldots, u_m) : \overline{\Omega} \to R^m$, we define $L_i u_i$ by

$$(L_i u_i)(x) = \sum_{j,k=1}^{N} a^i_{jk}(x)\partial^2 u_i/\partial x_j \partial x_k + \sum_{k=1}^{N} b^i_k(x)\partial u_i/\partial x_k$$

where $a^i_{jk} = a^i_{kj}$, $b^i_k \in C^{\lambda}(\Omega)$, $\lambda \in (0,1)$, $1 \leq j,\ k \leq N$, $1 \leq i \leq m$. Assume that $L_i u_i$ is uniformly elliptic, i.e., there exist constants c_1 and c_2, $0 < c_1 < c_2$ such that for all $\xi = (\xi_1, \ldots, \xi_m)$ and $x \in \Omega$, we have

$$c_1|\xi|^2 \leq \sum_{j,k=1}^{N} a^i_{jk}(x)\xi_j\xi_k \leq c_2|\xi|^2, \qquad 1 \leq i \leq m.$$

Let B_i be the boundary operator defined by

$$B_i u_i = a_i(x)\partial u_i/\partial\nu + d_i(x)u_i, \qquad x \in \partial\Omega,$$

where $a_i \geq 0$, $d_i \geq 0$ with $a_i + d_i > 0$ on $\partial\Omega$, and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. For $u = (u_1, \ldots, u_m) \in R^m$, $p^i = (p_{i1}, \ldots, p_{im}) \in R^m$, $1 \leq i \leq m$. We assume that $\Phi_i : \Omega \times R^m \times R^m \to R$, $(x, u, p^i) \mapsto \Phi_i(x, u, p^i)$, be uniformly Hölder continuous in x with the exponent λ, $0 < \lambda < 1$ and locally Lipschitz continuous in u and p^i. We shall consider the boundary value problem

$$-L_i u_i = \Phi_i(x, u, K^i u), \qquad x \in \Omega,\ 1 \leq i \leq m, \tag{2.1}$$

$$B_i u_i = h_i, \qquad x \in \partial\Omega,\ 1 \leq i \leq m, \tag{2.2}$$

where $K^i u = (K^{i1}, \ldots, K^{im})$. Each K^{ij} is a certain integral operator. In particular, it may be of the form $\int_{\Omega} \alpha(x, x', u(x'))dx'$, where α is locally Lipschitz continuous functions in $\Omega \times \Omega \times R^m$. By a solution of (2.1)-(2.2) we mean that a function $u(x)$ is continuous in $\overline{\Omega}$, having continuous derivatives up to second order in Ω and satisfies (2.1)-(2.2).

For reader's convenience we give two lemmas concerning the maximum principle, the existence-uniqueness and a priori estimates of the solution for the linear equation. Consider the boundary value problem for single equation

$$-Lu + \mu u = F(x) \qquad x \in \Omega,\ \mu > 0;\ Bu = h,\ x \in \partial\Omega, \tag{2.3}$$

where

$$Lu = \sum_{j,k=1}^{N} a_{jk}(x)\partial^2 u/\partial x_j \partial x_k + \sum_{k=1}^{N} b_k(x)\partial u/\partial x_k.$$

Lemma 2.1.([8]) *If the function u satisfying* (2.3) *with $F \geq 0$, $h \geq 0$, then $u(x) \geq 0$ in $\overline{\Omega}$.*

Lemma 2.2.([1]) *If $F \in C^{\lambda}(\Omega)$, and $h \in C^{1+\lambda}(\partial\Omega)$, then there exists a unique solution $u \in C^{2+\lambda}(\overline{\Omega})$ of* (2.3) *such that*

$$|u|_{C^{2+\lambda}(\overline{\Omega})} \leq \text{Const.}\left(|F|_{C^{\lambda}(\Omega)} + |h|_{C^{1+\lambda}(\partial\Omega)}\right).$$

3. Existence

In this section we shall consider the existence of the solution of the problem (2.1)-(2.2) by using Schauder fixed point theorem. Assume that

A1:

there exists a function $f : \overline{\Omega} \times R^m \times R^m \times R^m \times R^m \to R^m$, $(x, v, p, w, q) \mapsto f(x, v, p, w, q)$, $f = (f_1, \ldots, f_m)$ which satisfies the following conditions:

(3.1) $f(x, v, p, w, q)$ is uniformly Hölder continuous with the exponent λ, $0 < \lambda < 1$, in the domain Ω for v, p, w, q fixed and locally Lipschitz continuous with respect to v, p, w and q.

(3.2) There exist two functions $\alpha(x)$, $\beta(x) \in C^2(\overline{\Omega})$ and a positive constant ϵ such that

(i) $\alpha_i(x) \leq \beta_i(x)$, $x \in \overline{\Omega}$, $1 \leq i \leq m$,

(ii) $-L_i\alpha_i - f_i(x, v, K^i v, w, K^i w) \leq (v_i + w_i - 2\alpha_i)/2\epsilon$, $x \in \Omega$, $B_i\alpha_i(x) \leq h_i(x)$, $x \in \partial\Omega$.

(iii) $-L_i\beta_i - f_i(x, w, K^i w, v, K^i v) \geq (v_i + w_i - 2\beta_i)/2\epsilon$, $x \in \Omega$, $B_i\beta_i(x) \geq h_i(x)$, $x \in \partial\Omega$.

for all $v, w \in C(\Omega)$ with $\alpha_i \leq v_i \leq \beta_i$, $\alpha_i \leq w_i \leq \beta_i$ in Ω, $1 \leq i \leq m$.

(3.3) $f_i(x, v, K^i v, v, K^i v) = \Phi_i(x, v, K^i v)$, $1 \leq i \leq m$.

(3.4) If there exist functions v and w with $v_i \leq w_i$ in $\overline{\Omega}$, $1 \leq i \leq m$, and satisfying

$$-L_i v_i - f_i(x, v, K^i v, w, K^i w) = (w_i - v_i)/2\epsilon,\ x \in \Omega$$

$$-L_iw_i - f_i(x, w, K^iw, v, K^iv) = (v_i - w_i)/2\epsilon,\ x \in \Omega$$
$$B_iv_i(x) = B_iw_i(x) = h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m.$$

then $v \equiv w$ in $\overline{\Omega}$.

Remark: Such a function f is called a satellite of Φ.

A2:

Each operator $K^i(1 \leq i \leq m)$ defined in $N(\alpha, \beta) \cap C^\lambda(\overline{\Omega})$, taking values in $C^\lambda(\overline{\Omega})$, is bounded and continuous and has a bounded Fréchet derivatives on this set, here $N(\alpha, \beta) = \{u \in C(\overline{\Omega}) | \alpha_i \leq u_i \leq \beta_i \text{ in } \overline{\Omega},\ 1 \leq i \leq m\}$.
Let

$$V(\alpha, \beta) = \{u \in C^\lambda(\overline{\Omega}) | H^\lambda(u_i) \leq Q,\ \alpha_i \leq u_i \leq \beta_i \text{ in } \overline{\Omega},\ 1 \leq i \leq m\},$$

here $H^\lambda(u_i)$ denotes Hölder constant of u_i and Q is some constant to be determined later. For $v, w \in V(\alpha, \beta)$, we define

$$F_i(v, w)(x) = f_i(x, v(x), K^iv(x), w(x), K^iw(x)),\ x \in \overline{\Omega},\ 1 \leq i \leq m.$$

By using (3.1) and A2, we have the following result.

Lemma 3.1. *For $v, w \in V(\alpha, \beta)$, F_i $(1 \leq i \leq m)$ take values in $C^\lambda(\overline{\Omega})$ and satisfy Lipschitz condition, i.e., for $v, w, v^*, w^* \in V(\alpha, \beta)$ and $1 \leq i \leq m$, we have*

$$|F_i(v, w) - F_i(v^*, w^*)|_{C^\lambda(\overline{\Omega})} \leq const. \left(|v - v^*|_{C^\lambda(\overline{\Omega})} + |w - w^*|_{C^\lambda(\overline{\Omega})}\right).$$

For given a pair of functions $\eta, \tau \in V(\alpha, \beta)$, we define an operator $\mathcal{T}$ by

$$\mathcal{T}[\eta, \tau] \equiv [\mathcal{T}_1[\eta, \tau], \mathcal{T}_2[\eta, \tau]] = [v, w]$$

where v and w are the solutions of the problem

$$\begin{aligned} -L_iv_i + v_i/\epsilon &= F_i(\eta, \tau) + (\eta_i + \tau_i)/2\epsilon \text{ in } \Omega, \\ B_iv_i(x) &= h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} -L_iw_i + w_i/\epsilon &= F_i(\tau, \eta) + (\eta_i + \tau_i)/2\epsilon \text{ in } \Omega, \\ B_iw_i(x) &= h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m, \end{aligned} \tag{3.6}$$

The existence and uniqueness of v and w in $C^{2+\lambda}(\overline{\Omega})$ are guaranted by lemma 2.1 and lemma 2.2. Note that $\mathcal{T}_1[\tau, \eta] = \mathcal{T}_2[\eta, \tau]$.

Lemma 3.2. *The operator $\mathcal{T}$ carries $V(\alpha, \beta) \times V(\alpha, \beta)$ into itself.*

Proof. From (3.5) and (3.2)-(ii) with $v = \eta$ and $w = \tau$, we have

$$-L_i(v_i - \alpha_i) + (v_i - \alpha_i)/\epsilon \geq 0 \text{ in } \Omega,$$
$$B_i(v_i - \alpha_i)(x) = 0,\ x \in \partial\Omega.$$

By lemma 2.1, we have $v_i - \alpha_i \geq 0$ in $\overline{\Omega}$, $1 \leq i \leq m$. Similarly, we get $w_i - \beta_i \leq 0$ in $\overline{\Omega}$, $1 \leq i \leq m$. Next we claim that $H^\lambda(u_i) \leq Q$, $1 \leq i \leq m$. Let z be the solution of the problem

$$- L_i z_i + z_i/\epsilon = 0 \text{ in } \Omega,$$
$$B_i z_i(x) = h_i(x), \ x \in \partial\Omega.$$

Define $U_i = v_i - z_i$, $1 \leq i \leq m$, then U_i is a solution of the problem

$$- L_i U_i + U_i/\epsilon = F_i(\eta, \tau) + (\eta_i + \tau_i)/2\epsilon \text{ in } \Omega,$$
$$B_i U_i(x) = 0, \ x \in \partial\Omega, \ 1 \leq i \leq m,$$

By lemma 2.2, we obtain

$$|U_i|_{C^\lambda(\overline{\Omega})} \leq C_1 |F_i(\eta, \tau) + (\eta_i + \tau_i)/2\epsilon|_{C(\overline{\Omega})} \leq Q_1.$$

Thus we have

$$|v_i|_{C^\lambda(\overline{\Omega})} \leq |z_i|_{C^\lambda(\overline{\Omega})} + Q_1 \leq C_2 |h_i|_{C^{1+\lambda}(\partial\Omega)} + Q_1 \equiv Q,$$

Thus $H^\lambda(v_i) \leq Q$, $1 \leq i \leq m$. Similary, we also have $H^\lambda(w_i) \leq Q$, $1 \leq i \leq m$.

Lemma 3.3. *The operator $\mathcal{T}$ is compact and continuous. Furthermore, there exist $\overline{v}, \overline{w} \in C^{2+\lambda}(\overline{\Omega})$ such that $\mathcal{T}[\overline{v}, \overline{w}] = [\overline{v}, \overline{w}]$.*

Proof. For $v, w, v^*, w^* \in V(\alpha, \beta)$, $\mathcal{T}[v, w]$ and $\mathcal{T}[V^*, w^*]$ are in $C^{2+\lambda}(\overline{\Omega})$. Let $z = \mathcal{T}_1[v, w] - \mathcal{T}_1[v^*, w^*]$, then by lemma 2.2 and lemma 3.1, there exists a constant C_3 such that

$$|z_i|_{C^{2+\lambda}(\overline{\Omega})} \leq C_3 \left(|v - v^*|_{C^\lambda(\overline{\Omega})} + |w - w^*|_{C^\lambda(\overline{\Omega})} \right).$$

We also obtained similar inequality as above for $y = \mathcal{T}_2[v, w] - \mathcal{T}_2[v^*, w^*]$. Thus we have

$$|\mathcal{T}[v, w] - \mathcal{T}[v^*, w^*]|_{C^{2+\lambda}(\overline{\Omega})} \leq C_4 \left(|v - v^*|_{C^\lambda(\overline{\Omega})} + |w - w^*|_{C^\lambda(\overline{\Omega})} \right).$$

Therefore, $\mathcal{T}$ is continuous from $\mathcal{C}^\lambda(\overline{\Omega})$ into $\mathcal{C}^{2+\lambda}(\overline{\Omega})$. But $\mathcal{C}^{2+\lambda}(\overline{\Omega})$ is compact embeded in $\mathcal{C}^\lambda(\overline{\Omega})$. It follows that $\mathcal{T}$ is compact from $\mathcal{C}^\lambda(\overline{\Omega})$ into itself. By Schauder fixed point theorem, there exist $[\overline{v}, \overline{w}] \in V(\alpha, \beta) \times V(\alpha, \beta)$ such that $\mathcal{T}[\overline{v}, \overline{w}] = [\overline{v}, \overline{w}]$. Furthermore, $\overline{v}, \overline{w} \in C^{2+\lambda}(\overline{\Omega})$.

Theorem 3.4. *Assume that A1 and A2 are satisfied, then the problem* (2.1)-(2.2) *has a solution u with $\alpha_i \leq u_i \leq \beta_i$ in $\overline{\Omega}$, $1 \leq i \leq m$.*

Proof. From lemma 3.3, there exist $\overline{v}, \overline{w} \in V(\alpha, \beta) \cap C^{2+\lambda}(\overline{\Omega})$ such that $\mathcal{T}[\overline{v}, \overline{w}] = [\overline{v}, \overline{w}]$. In other words, they satisfy (3.5) and (3.6) with η, τ replaced by $\overline{v}$, $\overline{w}$ respectively. By (3.4) we get $\overline{v} \equiv \overline{w}$ in $\overline{\Omega}$, and by (3.3), we see that $u \equiv \overline{v} \equiv \overline{w}$ is a solution of the problem (2.1)-(2.2).

In the following, we shall give the sufficient conditions for the existence of the function f in A1. Assume that

B1: $\Phi(x,u,p): \Omega\times R^m\times R^m \to R^m$ is uniformly Hölder continous with the exponent λ, $0<\lambda<1$, in Ω for u, p fixed and locally Lipschitz continuous with respect to u and p.

B2: There exist functions $\alpha, \beta \in C^2(\overline{\Omega})$ and a positive constant ϵ such that

$$\begin{aligned}
&\text{(i) } \alpha_i(x) \le \beta_i(x),\ x\in\overline{\Omega},\ 1\le i\le m,\\
&\text{(ii) } -L_i\alpha_i - \Phi_i(x,v,K^iv) \le (v_i-\alpha_i)/\epsilon,\ x\in\Omega,\\
&\qquad B_i\alpha_i(x)\le h_i(x),\ x\in\partial\Omega.\\
&\text{(iii) } -L_i\beta_i - \Phi_i(x,v,K^iv) \ge (v_i-\beta_i)/\epsilon,\ x\in\Omega,\\
&\qquad B_i\beta_i(x)\ge h_i(x),\ x\in\partial\Omega.
\end{aligned}$$

for all $v\in C(\overline{\Omega})$ with $\alpha_i\le v_i\le\beta_i$ in $\overline{\Omega}$, $1\le i\le m$.

Remark. The functions α and β are called ϵ-lower and ϵ-upper solution of the problem (2.1)-(2.2). Define

$$f_i(x,v,p,w,q) = [\Phi_i(x,v,p)+\Phi_i(x,w,q)]/2.$$

Obviously, we have $f_i(x,v,K^iv,v,K^iv)=\Phi_i(x,v,K^iv)$ in Ω. For $\alpha_i\le v_i,\ w_i\le\beta_i$ in $\overline{\Omega}$, $1\le i\le m$. By B2, it is easy to see that (3.2) holds. It remains to show that (3.4) holds. Suppose that there exist function v and w with $v_i\le w_i$ in $\overline{\Omega}$, $1\le i\le m$ and satisfying those equalities in (3.4). Let $z=w-v$ in $\overline{\Omega}$, then

$$-L_iz_i + z_i/\epsilon = 0 \text{ in } \Omega, \quad B_iz_i(x)=0,\ x\in\partial\Omega.$$

By lemma 2.1, $z_i=0$ in $\overline{\Omega}$, i.e., $v_i=w_i$ in $\overline{\Omega}$, $1\le i\le m$. Therefore, we have the following theorem.

Theorem 3.5. *Assume that B1, B2 and A2 hold. Then the problem* (2.1)-(2.2) *has a solution $u(x)$ with $\alpha_i\le u_i\le\beta_i$ in $\overline{\Omega}$, $1\le i\le m$.*

The construction of ϵ-lower and ϵ-upper solutions can be easily obtained in the following example.

Example 3.6. Consider the boundary value problem for the steady state equation in [4, 6]:

$$\begin{aligned}
&-\nabla\cdot(D_1\nabla u) = -au - c_1G(v)u + q_1(x),\\
&-\nabla\cdot(D_2\nabla v) = -bv + c_2G(v)u + q_2(x), \quad \text{in } \Omega,\\
&Bu(x)=h_1(x),\ Bv(x)=h_2(x),\ x\in\partial\Omega,
\end{aligned} \tag{3.9}$$

where

$$G(v)(x) = \int_\Omega k(x,x')v(x')dx',$$

Assume that a, b, c_1 and c_2 are positive constants, that q_1 and q_2 are nonnegative functions in Ω, that D_1 and D_2 are positive functions in Ω, that k is a positive continuous function in $\Omega \times \Omega$, and that h_1 and h_2 are nonnegative functions on $\partial\Omega$. Let

$$k^*(x) = \int_\Omega k(x,x')dx',$$

Furthermore, we assume that $b^2 > 4q_2k^*$ in Ω and $2q_1c_2k^* < ab$ in Ω. Choose M such that

$$\max\{q_1/a,\ h_1/d_1,\ h_2/d_2\} < M < b/(2c_2k^*).$$

Set $\beta_1 = \beta_2 = M$ and $\alpha_1 = \alpha_2 = 0$, then we see that $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are ϵ-lower and ϵ-upper solutions of (3.9). By theorem 3.5, there exists a solution (u, v) of (3.9) such that $0 \le u \le M$, $0 \le v \le M$ in $\overline{\Omega}$. In particular, when $q_1 = 0$, $h_1 = h_2 = 0$, M is chosen to satisfy $0 < M < b/(2c_2k^*)$.

Example 3.7. Consider the steady state equation for the system in predator-prey dynamics from [9]:

$$\begin{aligned} &-\nabla\cdot(D_1\nabla u) = u(a - bu - cR_1(v)) \\ &-\nabla\cdot(D_2\nabla v) = v(-d - ev + fR_2(u)), \quad \text{in } \overline{\Omega}, \\ &Bu(x) = h_1(x) \ge 0,\ Bv(x) = h_2(x) \ge 0,\ x \in \partial\Omega, \end{aligned} \tag{3.10}$$

where

$$R_k(\psi)(x) = \int_{-\infty}^0 \int_\Omega G_k(x,y)\psi(s,y)dy\, F_k(s)\, ds,\ k = 1, 2.$$

and a, b, c, d, e, f are positive constants and G_k, F_k are smooth nonnegative functions. An ϵ-upper solution β and an ϵ-lower solution α are given by setting $\beta_1 = \beta_2 = M$ and $\alpha_1 = \alpha_2 = 0$, where M is to be determined later. Let

$$C^* = \max_k \max_{x\in\overline{\Omega}} \left(\int_{-\infty}^0 F_k(s)\, ds \cdot \int_\Omega G_k(x,y)dy \right) > 0.$$

(i) if $fC^* \le e$, then we may choose M large enough.
(ii) if $fC^* > e$ and $E = \max\{b/a,\ h_1/d_1,\ h_2/d_2\} < d/(fC^* - e)$, then we choose M such that $E < M < d/(fC^* - e)$.
By theorem 3.5, problem (3.10) has a solution (u, v) such that $0 \le u \le M$, $0 \le v \le M$ in $\overline{\Omega}$.

Remark. Another different type of systems involving nonlinear integral operators is also given in [10].

4. Modified method of mixed monotony

In this section we shall give a constructive proof for the existence of solutions of (2.1)-(2.2) by finding two monotone sequences which converge uniformly in $\overline{\Omega}$ to the solution. Assume that

C1: there exists a function $f : \Omega \times R^m \times R^m \times R^m \times R^m \to R^m$, $(x, v, p, w, q) \mapsto f(x, v, p, w, q)$, $f = (f_1, \dots, f_m)$ which satisfies the following conditions:
(4.1) $f(x, v, p, w, q)$ is uniformly Hölder continuous with the exponent λ, $0 < \lambda < 1$, in the domain Ω for v, p, w, q fixed and locally Lipschitz continuous with respect to v, p, w and q. Furthermore, for some $\epsilon > 0$, we have

$$\begin{aligned} &f_i(x, v, p, w, q) - f_i(x, v, p, w^*, q^*) \geq 0, \\ &f_i(x, v, p, w, q) - f_i(x, v^*, p^*, w, q) + (v_i - v_i^*)/\epsilon \geq 0, \end{aligned} \tag{$*$}$$

if $v_i \geq v_i^*$, $p_i \geq p_i^*$, $w_i \leq w_i^*$, $q_i \leq q_i^*$, $1 \leq i \leq m$.
(4.2) There exist two functions $\alpha(x), \beta(x) \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} &\text{(i) } \alpha_i(x) \leq \beta_i(x),\ x \in \overline{\Omega},\ 1 \leq i \leq m, \\ &\text{(ii) } -L_i\alpha_i - f_i(x, \alpha, K^i\alpha, \beta, K^i\beta) \leq 0,\ x \in \Omega, \\ &\qquad B_i\alpha_i(x) \leq h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m. \\ &\text{(iii) } -L_i\beta_i - f_i(x, \beta, K^i\beta, \alpha, K^i\alpha) \geq 0,\ x \in \Omega, \\ &\qquad B_i\beta_i(x) \geq h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m. \end{aligned}$$

(4.3) $f_i(x, v, K^iv, v, K^iv) = \Phi_i(x, v, K^iv)$, $1 \leq i \leq m$.
(4.4) If there exist functions v and w with $v_i \leq w_i$ in $\overline{\Omega}$, $1 \leq i \leq m$, and satisfying

$$\begin{aligned} &-L_iv_i - f_i(x, v, K^iv, w, K^iw) = 0, \\ &-L_iw_i - f_i(x, w, K^iw, v, K^iv) = 0 \text{ in } \Omega, \\ &B_iv_i(x) = B_iw_i(x) = h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m. \end{aligned} \tag{$*$}$$

then $v \equiv w$ in $\overline{\Omega}$.

C2: Each operator $K^i (1 \leq i \leq m)$ defined in $N(\alpha, \beta) \cap C^\lambda(\overline{\Omega})$, taking values in $C^\lambda(\overline{\Omega})$, is bounded and continuous and has a bounded Fréchet derivatives on this set, here $N(\alpha, \beta) = \{u \in C(\overline{\Omega}) | \alpha_i \leq u_i \leq \beta_i \text{ in } \overline{\Omega},\ 1 \leq i \leq m\}$. Furthermore, we assume that K^i is nondecreasing in u.

We then have the following theorem:

Theorem 4.1. *Assume that C1 and C2 hold. Then there exist two monotone sequences $\{u_n(x)\}$ and $\{u^n(x)\}$ which converge to a solution $u(x)$ of the problem (2.1)-(2.2) and*

$$\alpha \leq u_1 \leq u_2 \leq \dots \leq \underline{u} = u = \overline{u} \leq \dots \leq u^2 \leq u^1 \leq \beta \qquad \text{in } \overline{\Omega}.$$

Proof. The sequence $\{u_n(x)\}$ and $\{u^n(x)\}$, $n \geq 0$, are defined as follows:

$$- L_i u^n{}_i + u^n{}_i/\epsilon = F_i(u^{n-1}, u_{n-1}) + u^{n-1}{}_i/\epsilon \text{ in } \Omega, \qquad (4.5)$$
$$B_i u^n{}_i(x) = h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m,$$

and

$$- L_i u_{n_i} + u_{n_i}/\epsilon = F_i(u_{n-1}, u^{n-1}) + u_{n-1_i}/\epsilon \text{ in } \Omega, \qquad (4.6)$$
$$B_i u_{n_i}(x) = h_i(x),\ x \in \partial\Omega,\ 1 \leq i \leq m,$$

and

$$u^0(x) = \beta(x),\ u_0(x) = \alpha(x),\ x \in \overline{\Omega}. \qquad (4.7)$$

First we show that $\{u^n\}$ is a monotone nonincreasing sequence and $\{u_n\}$ is a monotone nondecreasing sequence. In fact, by (4.5) and (4.2)(iii), putting $z_i = \beta_i - u^1{}_i$, $1 \leq i \leq m$, we have

$$-L_i z_i + z_i/\epsilon \geq 0 \text{ in } \Omega, \quad B_i z_i(x) \geq 0,\ x \in \partial\Omega.$$

By lemma 2.1, $z_i \geq 0$, i.e., $\beta_i \geq u^1{}_i$ in Ω, $1 \leq i \leq m$. Similarly, we get $\alpha_i \leq u_{1_i}$ in $\overline{\Omega}$, $1 \leq i \leq m$, Then by induction proof on n. Let $z = u^n - u^{n+1}$, by (4.1)(*), we have

$$- L_i z_i + z_i/\epsilon = F_i(u^{n-1}, u_{n-1}) - F_i(u^n, u_n) + (u^{n-1}{}_i - u^n{}_i)/\epsilon \geq 0 \text{ in } \Omega,$$
$$B_i z_i(x) = 0,\ x \in \partial\Omega.$$

By lemma 2.1, we have $z_i \geq 0$ in $\overline{\Omega}$, $1 \leq i \leq m$. Hence $u^n \geq u^{n+1}$ in $\overline{\Omega}$. Similarly, we have $u_n \leq u_{n+1}$ and $u_n \leq u^n$ in $\overline{\Omega}$ for $n \geq 0$. Thus we obtain

$$\alpha \leq u_1 \leq u_2 \leq \ldots \leq u_{n-1} \leq u_n \leq \ldots \leq u^n \leq u^{n-1} \leq \ldots \leq u^2 \leq u^1 \leq \beta \text{ in } \overline{\Omega}.$$

Let

$$\underline{u}(x) = \lim_{n\to\infty} u_n(x), \quad x \in \overline{\Omega}$$
$$\overline{u}(x) = \lim_{n\to\infty} u^n(x), \quad x \in \overline{\Omega},$$

then $\underline{u} \leq \overline{u}$ in $\overline{\Omega}$ and $u_n \to \underline{u}$, $u^n \to \overline{u}$ in the norm of $L^p(\Omega)$, $p \geq 1$. It follows that u^n and u_n converge in $W^{1,p}(\Omega)$, $1 < p < \infty$. By Sobolev embedding theorem, u^n and u_n converge in $C^\lambda(\overline{\Omega})$ for $\lambda = 1 - \frac{N}{p}$. By lemma 2.2, u^n and u_n converge in $C^{2+\lambda}(\overline{\Omega})$. Thus we have (4.4)(*) with $v = \underline{u}$ and $w = \overline{u}$. By (4.4), $\overline{u} \equiv \underline{u}$ in $\overline{\Omega}$.

A sufficient condition for the existence of the function f in C1 is guaranteed by imposing the following assumptions;

D1: $\Phi(x, u, p) : \Omega \times R^m \times R^m \to R^m$ is uniformly Hölder continuous with the exponent λ, $0 < \lambda < 1$, in Ω for u, p fixed and locally Lipschitz continuous with respect to u and p. Furthermore, there exists a positive constant ϵ such that

$$\Phi_i(x, w, K^i w) - \Phi_i(x, v, K^i v) \geq 2(v_i - w_i)/\epsilon \qquad (4.8)$$

for $\alpha_i \le v_i \le w_i \le \beta_i$, $1 \le i \le m$. And

D2: There exist functions $\alpha, \beta \in C^2(\overline{\Omega})$ such that

$$\begin{aligned} &\text{(i) } \alpha_i(x) \le \beta_i(x),\ x \in \overline{\Omega},\ 1 \le i \le m, \\ &\text{(ii) } -L_i\alpha_i - \Phi_i(x, \alpha, K^i\alpha) \le (\beta_i - \alpha_i)/\epsilon,\ x \in \Omega, \\ &\qquad B_i\alpha_i(x) \le h_i(x),\ x \in \partial\Omega. \\ &\text{(iii) } -L_i\beta_i - \Phi_i(x, \beta, K^i\beta) \ge (\alpha_i - \beta_i)/\epsilon,\ x \in \Omega, \\ &\qquad B_i\beta_i(x) \ge h_i(x),\ x \in \partial\Omega,\ 1 \le i \le m. \end{aligned}$$

Define

$$f_i(x, v, p, w, q) = [\Phi_i(x, v, p) + \Phi_i(x, w, q)]/2 + 2(w_i - v_i)/\epsilon,\ 1 \le i \le m. \tag{4.9}$$

Then (4.3) holds. By (4.8) and D2, we see that (4.2) is satisfied. It is easy to see that (4.1)(*) is satisfied by using (4.9) and that (4.4) holds. Therefore, we have the following theorem:

Theorem 4.2. *Assume that D1, D2 and C2 are satisfied, then there exist two monotone sequences* $\{u_n(x)\}$ *and* $\{u^n(x)\}$ *which converge to a solution* $u(x)$ *of the problem* (2.1)-(2.2) *and*

$$\alpha \le u_1 \le u_2 \le \ldots \le u \le \ldots \le u^2 \le u^1 \le \beta \quad \text{in } \overline{\Omega}.$$

References

[1] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order", *Springer Verlag*, 1983.

[2] M. Khavanin, "The method of mixed monotony and second order integro-differential system", *Appl. Anal.* 28 (1988), 199-206.

[3] M. Khavanin and V. Lakshmikantham, "The method of mixed monotony and second order boundary value problems", *JMAA*, 120 (1986), 737-744.

[4] P. de. Mottoni, E. Orlandi and A. Tesei, "Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection", *Nonlinear Analysis, TMA*, 3 (1979), 663-675.

[5] V.I. Opoitsev, "Generalization of the theory of monotone and concave operators", *Tr. Mosk. Mathem*, 36 (1978), 237-273.

[6] C.V. Pao, "On nonlinear reaction-diffusion systems", *JMAA*, 87 (1982), 165-198.

[7] V.P. Politjukov, "On the theory of upper and lower solutions and the solvability of quasilinear integro-differential equations", *Math. USSR, Sbornik*, 35 (1979), 499-507.

[8] M.H. Protter and H.F. Weinberger, "Maximum Principles in Differential Equations", *Prentice-Hall, Englewood, Cliffs, N.J.*, 1967.

[9] R. Redlinger, "Lower and upper solution for strongly coupled systems of reaction diffusion equationss, Proceeding of inter national conference on theory and application of differential equations", *Ohio Univ.*, (1988), 327-332.

[10] L.Y. Tsai, "On the solvability of nonlinear integro-differential operators", *Chinese J. Math.*, (1983), 75-84.

Department of Mathematical Sciences, National Chengchi University, Taipei, Taiwan, R.O.C. 11623.