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SUMANIFOLDS OF EUCLIDEAN SPACES SATISFYING $\Delta H = AH$

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Abstract. In [5] the author initiated the study of submanifolds whose mean curvature vector H satisfying the condition $\Delta H = \lambda H$ for some constant λ and proved that such submanifolds are either biharmonic or of 1-type or of null 2-type. Submanifolds of hyperbolic spaces and of de Sitter space-times satisfy this condition have been investigated and classified in [6,7]. In this article, we study submanifolds of E^m whose mean curvature vector H satisfies a more general condition; namely, $\Delta H = AH$ for some $m \times m$ matrix A.

1. Introduction

Let E_s^m be the m-dimensional pseudo-Euclidean space with the standard flat metric given by

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2, \tag{1.1}$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of E_s^m. For a positive number r, we put

$$S_s^{m-1}(r) = \{x \in E_s^m :< x, x >= r^2\}, \tag{1.2}$$

$$H_{s-1}^{m-1}(-r) = \{x \in E_s^m :< x, x >= -r^2\}, \tag{1.3}$$

where $<,>$ denotes the indefinite inner product on the pseudo-Euclidean space. In physics, $S_1^{m-1}(r)$ and E_1^m are known as a *de Sitter space-time* and the Minkowski space-time, respectively. We denote by $H^{m-1}(-r)$ the (connected) *hyperbolic space*, imbedded standardly in E_1^m, by

$$H^{m-1}(-r) = \{x \in E_1^m :< x, x >= -r^2 \text{ and } x_1 > 0\}. \tag{1.4}$$

Let $x : M \to E_s^m$ be an isometric immersion from a pseudo-Riemannian manifold M into E_s^m. Denote the position vector field of the immersion $x : M \to E_s^m$, also by x. Then we have

$$\Delta x = -nH, \tag{1.5}$$

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where H is the mean curvature vector of M in E_s^m. It follows immediately from (1.5) that M is minimal in E_s^m if and only if the immersion x is harmonic, *i.e.*, $\Delta x = 0$. The immersion $x : M \to E_s^m$ is called *biharmonic* if $\Delta^2 x = 0$. A submanifold of E_s^m is biharmonic if and only if its mean curvature vector H is an eigenvector of Δ with eigenvalue 0.

The study of submanifolds whose mean curvature vector is an eigenvector of Δ was initiated in [5] in which the author proved that a submanifold M of a Euclidean space satisfies the condition $\Delta H = \lambda H$ for some constant λ if and only if either M is biharmonic ($\lambda = 0$), or M is of 1-type or of null 2-type. The complete classification of sufaces in E^3 satisfy the condition $\Delta H = \lambda H$ was done in 1988 (cf. [6]).

In [7,8], the author investigated submanifolds of hyperbolic spaces and of de Sitter space-time satisfying the condition $\Delta H = \lambda H$. Among others, he proved the following results.

Theorem A. *Let M be an n-dimensional submanifold of the hyperbolic space H^{m-1} (-1), imbedded standardly in the Minkowski space-time E_1^m. Then the mean curvature vector H of M in E_1^m is an eigenvector of Δ if and only if either M is a minimal submanifold of $H^{m-1}(-1)$ or M is contained in a totally umiblical hypersurface of $H^{m-1}(-1)$ as minimal submanifold.*

Theorem B. *Let M be an n-dimensional, non-minimal submanifold of the hyperbolic space $H^{m-1}(-1)$, imbedded standardly in the Minkowski space-time E_1^m. Then the following statements are equivalent:*
(a) *M is a biharmonic submanifold of E_1^m;*
(b) *The mean curvature vector of M in E_1^m is a light-like constant vector;*
(c) *M is a pseudo-umbilical submanifold with unit parallel mean curvature vector in $H^{m-1}(-1)$;*
(d) *M is contained in a flat totally umbilical hypersurface of $H^{m-1}(-1)$ as a minimal submanifold.*

Theorem C. *Let M be an n-dimensional, space-like submanifold of the de Sitter space-time $S_1^{m-1}(1)$, imbedded standardly in the Minkowski space-time E_1^m. Then the following statements are equivalent:*
(a) *M is a biharmonic submanifold of E_1^m;*
(b) *The mean curvature vector of M in E_1^m is a light-like constant vector;*
(c) *M is a pseudo-umbilical submanifold with unit (time-like) parallel mean curvature vector in $S_1^{m-1}(1)$;*
(d) *M is contained in a flat totally umbilical hypersurface of $S_1^{m-1}(1)$ as a minimal submanifold.*

In [1,10,12], surfaces in a Euclidean space whose Gauss map G satisfies the condition: $\Delta G = AG$ for some matrix A were studied. As a generalization of these conditions, we shall investigate in this article the following geometric problem.

Problem : *Study submanifolds in a Euclidean m-space E^m (or more generally in a pseudo-Euclidean space E_s^m) whose mean curvature vector H satisfies the condition: $\Delta H = AH$ for some matrix $A \in R^{m\times m}$.*

2. Some General Results

Let $x : M \to E^m$ be an immersion from an n-dimensional, connected manifold M into the Euclidean m-space E^m. With respect to Riemannian metric g on M induced from the Euclidean metric of the ambient space E^m, M is a Riemannian manifold. Denote by Δ the Laplacian operator of the Riemannian manifolds (M, g). By applying the Laplacian operator we have the notion of submanifolds of finite type introduced in [3](see also [6]). The immersion x is said to be of *finite type* if each component of the position vector field of M in E^m, also denoted by x, can be written as a finite sum of eigenfunctions of the Laplacian operator, that is, if

$$x = c + x_1 + x_2 + \ldots + x_k \tag{2.1}$$

where c is a constant vector, $x_1, \ldots, x_k$ are non-constant maps satisfying $\Delta x_i = \lambda_i x_i$, $i = 1, \cdots, k$. If in particular all eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ are mutually different, then the immersion x (or the submanifold M) is said to be of *k-type* and the decomposition (2.1) is called the *spectral decomposition* of the immersion x. A k-type immersion is said to be of null k-type if one of $\lambda_1, \ldots, \lambda_k$ is zero. An immersion x (or a submanifold M) is said to be of *infinite type* if it is not of finite type.

Lemma 1. *Let M be a submanifold of E^m whose mean curvature vector H satisfies $\Delta H = AH$ for some $m \times m$ matrix A. Then there is a polynomial $P(t)$ of degree $\leq m$ such that $P(\Delta)H = 0$,*

Proof. Let $P(t)$ be the minimal polynomial of A. Then P is a polynomial of degree $\leq m$ satisfying $P(A) = 0$ according to the Cayley-Hamilton theorem. Thus, $P(\Delta)H = P(A)H = 0$.

Combining Lemma 1 with a result of [3] we obtain immediately the following

Lemma 2. *Let M be a compact submanifold of E^m. If the mean curvature vector of M in E^m satisfying the condition $\Delta H = AH$ for some $m \times m$ matrix A, then M is of k-type with $k \leq m$. In particular, M is of finite type.*

If $x : M \to E^m$ is an immersion of null k-type, then, without loss of generality, we may assume that

$$x = x_1 + \ldots + x_k, \qquad \Delta x_i = \lambda_i x_i, \quad \lambda_1 = 0 \tag{2.2}$$

is the spectral decomposition of the immersion x.

For each $i \in \{1, \cdots, k\}$ we put $E_i = Span\{x_i(p) : p \in M\}$. Then each E_i is a linear subspace of E^m.

Definition 2.1. Let $x : M \to E^m$ be an immersion of k-type (respectively, of null k-type) whose spectral decomposition is given by (2.1) (respectively, by (2.2)). Then the immersion x is said to be *linearly independent* (respectively, *weakly linearly independent*) if the subspaces $E_1, \ldots, E_k$ (respectively, $E_2, \ldots, E_k$) are linearly independent. And the immersion x is said to be *orthogonal* (respectively, *weakly orthogonal*) if the subspaces $E_1, \ldots, E_k$ (respectively, $E_2, \ldots, E_k$) are mutually orthogonal in E^m.

If a k-type submanifold M is not null, then the notions of linearly independence and weakly linearly independence are the same.

Theorem 3. *Let M be a submanifold of finite type in a Euclidean m-space E^m. Then M satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A if anf only M is weakly linearly independent.*

Proof. If M is not null and the immersion is linearly independent, then the results follows from (1.5) and Theorem 2.2 of [10]. So, we suppose $x : M \to E^m$ is a weakly linearly independent immersion of null k-type whose spectral decomposition is given by (2.2). For each $i \in \{2, \cdots, k\}$ we choose a basis $\{c_{ij} : j = 1, \cdots, m_i\}$ of E_i, where m_i is the dimension of E_i. Put $\ell = m_1 + \ldots + m_k$ and E^ℓ denote the subspace of E^m spanned by $E_1, \ldots, E_k$. Since the immersion x is assumed to be weakly linearly independent, the vectors $\{c_{ij} : i = 2, \cdots, k; j = 1, \cdots, m_i\}$ are linearly independent ℓ vectors in E^ℓ. Furthermore, we choose the Euclidean coordinate system $(u_1, \ldots, u_m)$ on E^m such that E^ℓ is defined by $u_{\ell+1} = \cdots = u_m = 0$. Regard each c_{ij} as a column ℓ-vector. We put

$$S = (c_{21}, \ldots, c_{2m_1}, \ldots, c_{k1}, \cdots, c_{km_k}). \tag{2.3}$$

Then the matrix S is a nonsingular $\ell \times \ell$ matrix. Let D denote the diagonal $\ell \times \ell$ matrix given by

$$D = Diag(\lambda_1, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k), \tag{2.4}$$

where λ_i repeats m_i-times. If we put $A = SDS^{-1}$, then $Ac_{ij} = \lambda_i c_{ij}$ for any $i \in \{2, \cdots, k\}$ and $j \in \{1, \cdots, m_i\}$. Therefore, we have

$$\Delta y = Ay \tag{2.5}$$

for the map $y : M \to E^\ell$ given by $y = x_2 + \cdots + x_k = x - x_1$. By regarding the $\ell \times \ell$ matrix A as an $m \times m$ matrix in a natural way (with zeros for each of the additional entries), we obtain

$$\Delta x = A(x - x_1), \tag{2.6}$$

for the immersion $x : M \to E^m$. From (1.5), (2.2) and (2.6), we obtain $\Delta H = AH$.

Conversely, suppose M is a submanifold of finite type whose mean curvature vector H in E^m satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A. Then, by (1.5), we have $\Delta^2 x = A\Delta x$. Thus,

$$\Delta(\Delta x - Ax) = 0 \tag{2.7}$$

Put $\overline{x}_1 = Ax - \Delta x$.

Case (1): $\overline{x}_1 = 0$. In this case, $\Delta x = Ax$. Therefore the result follows from Theroem 2.2 of [10].

Case (2): $\overline{x}_1 \neq 0$. If M is not null, then

$$x = c + x_1 + \ldots + x_k, \qquad \Delta x_i = \lambda_i x_i, \quad \lambda_1, < \ldots < \lambda_k \tag{2.8}$$

for some constant vector c and non-constant maps $x_1, \ldots, x_k$.

From (1.5) and (2.8), we find

$$-nH = \lambda_1 x_1 + \ldots + \lambda_k x_k, \tag{2.9}$$

$$-n\Delta H = \lambda_1^2 x_1 + \ldots + \lambda_k^2 x_k. \tag{2.10}$$

Thus, by using the condition $\Delta H = AH$, (2.9) and (2.10), we get

$$\lambda_1(Ax_1 - \lambda_1 x_1) + \ldots + \lambda_k(Ax_k - \lambda_k x_k) = 0. \tag{2.11}$$

By applying Δ^i to (2.11), we may obtain

$$\lambda_1^i(Ax_1 - \lambda_1 x_1) + \ldots + \lambda_k^i(Ax_k - \lambda_k x_k) = 0, \quad i = 1, \cdots, k. \tag{2.12}$$

Since $\lambda_1, \ldots, \lambda_k$ are mutually distinct, (2.12) implies $Ax_i = \lambda_i x_i, i = 1, \cdots, k$. This shows that $x_1, \cdots, x_k$ are eigenvectors belonging to distinct eigenvalues. Consequently, the subspaces spaces $E_1, \ldots, E_k$ spanned by $x_1, \ldots, x_k$, respectively, are linearly independent. Therefore, the immersion is linearly independent.

If the immersion x is of null k-type, then we have

$$x = x_1 + \cdots + x_k, \qquad \Delta x_i = \lambda_i x_i, \quad \lambda_1 = 0. \tag{2.13}$$

Thus, by $Ax - \Delta x = \overline{x}_1$, we get

$$(Ax_1 - \overline{x}_1) + (Ax_1 - \lambda_1 x_1) + \ldots (Ax_k - \lambda_k x_k) = 0$$

from which we may obatin

$$Ax_1 = \overline{x}_1, \qquad Ax_i = \lambda_i x_i, \quad i = 2, \cdots, k.$$

Therefore, the subspaces $E_2, \ldots, E_k$ spanned by $x_2, \ldots, x_k$ are linearly independent. Hence, the immersion is weakly linearly independent.

In [11], Dillen, Pas and Verstraelen studied submanifolds of E^m satisfying the condition $\Delta x = Ax + B$ for some $m \times m$ matrix A and constant vector B. From (1.5), it follows that condition $\Delta x = Ax + B$ implies condition $\Delta H = AH$. So, it is natural to ask whether these two conditions are equivalent. In fact, these two conditions are equivalent when the submanifold is compact. However, if the submanifold is not compact, then the condition $\Delta H = AH$ is indeed weaker than Dillen-Pas-Verstraelen's condition. More precisely, we have the following.

Corollary 4. *If M is a compact submanifold of E^m, then the mean curvature vector of M in E^m satisfying the condition $\Delta H = AH$ for some $m \times m$ matrix A if and only if M satisfies Dillen-Pas-Verstraelen's condition.*

Proof. Suppose M is a compact submanifold which satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A. Then M is non-null. From the condition $\Delta H = AH$, we get $\Delta(\Delta x - Ax) = 0$. This implies $\Delta x - Ax$ is a constant vector, say B, since M is compact. So, $\Delta x = Ax + B$. The converse follows from discussion above.

Corollary 5. *Let M be a weakly linearly independent submanifold of null k-type in E^m whose position vector is given by (2.2). If $Span\{x_1\} \cap Span\{x_2, \ldots, x_k\} \neq \emptyset$, then M satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A, but it does not satisfy Dillen-Pas-Verstraelen's condition.*

Proof. Let M be a null k-type weakly linearly independent submanifold of E^m whose spectral decomposition is given by (2.2). Then, by Theorem 3, M satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A. On the other hand, since M is not null k-type and Span$\{x_1\}\cap$ Span$\{x_1, \ldots, x_k\} \neq \emptyset$, M is not linearly independent. Therefore, by Theorem 2.2 of [10], M does not satisfies Dillen-Pas-Verstraelen's coondition.

We need the following

Lemma 6. *Let $x : M \to E^m$ be an isometric immersion of finite type. If there is a polynomial Q of degree d such that $Q(\Delta)H = 0$, then x is of k-type with $k \leq d+1$. In particular, if x is of $(d+1)$-type, then x is of null $(d+1)$-type.*

Proof. If $M$is of k-type, the we have the spectral decomposition (2.1). By (1.5) and (2.1), we have

$$-n\Delta^j H = \lambda_1^{j+1}x_1 + \ldots + \lambda_k^{j+1}x_k, \qquad j = 0, 1, 2, \cdots \tag{2.14}$$

Thus, by $Q(\Delta)H = 0$, we find

$$\lambda_1 Q(\lambda_1)x_1 + \ldots + \lambda_k Q(\lambda_k)x_k = 0. \tag{2.15}$$

By applying Δ^j to (2.15), we obtain

$$\lambda_1^{j+1}Q(\lambda_1)x_1 + \ldots + \lambda_k^{j+1}Q(\lambda_k)x_k = 0, \qquad j = 0, 1, 2, \cdots. \tag{2.16}$$

Since $\lambda_1, \cdots, \lambda_k$ are mutually distinct, (2.16) implies that either M is of k-type with $k \leq d$ or M is of null $(d+1)$-type.

3. Ruled Surfaces Satisfying $\Delta H = AH$

In this section we completely classify ruled surfaces in E^m satisfying the condition $\Delta H = AH$ for some $m \times m$ matrix A.

Theorem 7. *A cylinder over a weakly linearly independent curve of finite type and part of a helicoid in an affine 3-space are the only ruled surfaces in E^m satisfying the condition $\Delta H = AH$ for some $m \times m$ matrix A.*

Proof. We consider two cases separately:

Case 1. M is a cylinder.

Suppose that the surface M is a cylinder over a curve γ in an affine hyperplane E^{n-1}, which we can choose to have the equation $x_n = 0$. Assume that γ is parametrized by its arc length s. Then a parametrization X of M is given by

$$x(s,t) = \gamma(s) + te_n. \tag{3.1}$$

The Laplacian Δ of M is given in terms of s and t by $\Delta = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}$ and the Laplacian Δ' of γ is given by $\Delta' = -\frac{\partial^2}{\partial s^2}$.

If the surface M satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A, then by Lemma 1, there exist real numbers $c_1, \ldots, c_m$ such that

$$\Delta^{m+1}x + c_1\Delta^m x + \cdots + c_m \Delta x = 0. \tag{3.2}$$

From (3.1) and (3.2) we get

$$\Delta'^{m+1}\gamma + c_1\Delta'^m\gamma + \cdots + c_m\Delta'\gamma = 0.$$

which implies that γ is of finite type by Proposition 4.1 of [10]. Therefore, the surface is a cylinder over a curve of finite type. Furthermore, because the surface is a cylinder, the condition $\Delta H = AH$ implies that $\Delta' H' = \overline{A}H'$, where H' is the mean curvatur vector of γ and $\overline{A}$ is the $(m-1) \times (m-1)$-matrix obtained from A, restricted to the hyperplane defined by $x_n = 0$. Hence, by applying Theorem 3, γ is a weakly linearly independent curve.

Case 2: M is not cylindrical.

If the ruled surface M is not cylindrical, we can decompose M into open pieces such that on each piece we find a parametrization x of the form:

$$x(s,t) = \alpha(s) + t\beta(s)$$

where α and β are curves in E^m such that

$$<\alpha', \beta>= 0, \qquad <\beta, \beta>= 1, \qquad <\beta', \beta'>= 1.$$

We have $x_s = \alpha' + t\beta'$ and $x_t = \beta$. We define functions q, u and v by

$$q = \|x_s\|^2 = t^2 + 2ut + v, \quad u =< \alpha', \beta' >, \quad v =< \alpha', \alpha' > . \tag{3.3}$$

The Laplacian Δ of M can be expressed as follows (cf. [9]):

$$\Delta = -\frac{\partial^2}{\partial t^2} - \frac{1}{q}\frac{\partial^2}{\partial s^2} + \frac{1}{2}\frac{\partial q}{\partial s}\frac{1}{q^2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial q}{\partial t}\frac{1}{\partial t}\frac{1}{q}\frac{\partial}{\partial t}. \tag{3.4}$$

From the Lemma of [9], we know that if Q is a polynomial in t with functions in s as coefficients and $\deg(Q) = d$, then

$$\Delta(\frac{Q(t)}{q^m}) = \frac{\widetilde{Q}(t)}{q^{m+3}}$$

where $\widetilde{Q}$ is a polynomial in t with functions in s as coefficients and $\deg(\widetilde{Q}) \leq d + 4$.

We suppose that the surface M satisfies the condition $\Delta H = AH$ for some $m \times m$ matrix A. Then, by Theorem 3, there exist real numbers $c_1, \ldots, c_m$ such that

$$\Delta^{m+1}x + c_1\Delta^m x + \cdots + c_m\Delta x = 0. \tag{3.5}$$

We know that every component of x is a linear function in t with functions in s as coefficients. By applying the Lemma of [9], we have

$$\Delta^r x = \frac{P_r(t)}{q^{3r-1}},$$

where P_r is a vector whose components are polynomials in t with functions in s as coefficients and $\deg(P_r) \leq 1+4r$. Hence, if r goes up by one, the degree of the numberator of any component of $\Delta^r x$ goes up by at most 4, while the degree of the denominator goes up by 6. Hence, the sum (3.5) can never be zero, unless of course $\Delta x = 0$. But then M is a minimal ruled surface. In this case, the result follows from the well-known classification of minimal ruled surfaces.

Corollary 8. *Minimal surface and circular cylinders are the only ruled surfaces in E^3 satisfying the condition $\Delta H = AH$ for some 3×3 matrix A.*

Proof. This corollary follows from the fact that lines and circles are the only curves of finite type in a plane (cf. [9]).

Corollary 9. *Minimal surfaces and circular cylinders are the only finite type surfaces in E^3 satisfying the condition $\Delta H = AH$ for some 3×3 singular matrix A.*

Proof. Suppose M is a surface in E^3 satisfying the condition $\Delta H = AH$ for some singular 3×3 matrix A. If $A = 0$, then $\Delta H = 0$, *i.e.*, M is a biharmonic surface in E^3. Hence, by a result of the author (cf. [6]), M is a minimal surface in E^3.

If $\operatorname{rank}(A) = 1$ and $H \neq 0$, then Lemma 6 implies that M is either of null 2-type or of 1-type. If M is of null 2-type, M is a part of circular cylinder. If M is of 1-type, M is a part of an ordinary sphere; in this case, A is a scalar multiple of the identity matrix which is a contradiction.

If $\operatorname{rank}(A) = 2$ and $H \neq 0$, then the minimal polynomial of A is of degree 2. In this case, M is either of 2-type or of null 3-type by Lemma 6. In either cases we have

$$-2H = \Delta x = \lambda_1 x_1 + \lambda_2 x_2, \tag{3.6}$$

for some distinct λ_1, λ_2.

On the other hand, from the proof of Theorem 3, we have $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2$. Thus, by (3.6), we know that H lies in the image subspace of A. Hence, if b is unit vector perpendicular to the image subspace of A, then b is a constant vector tangent to the surface. Therefore, M is a ruled surface in E^3. Hence, by applying Corollary 8, we conclude that M is a part of circular cylinder.

4. Tubes Satisfying $\Delta H = AH$

In this section we completely classify tubes in E^3 satisfying the condition $\Delta H = AH$.

Theorem 10. *Circular cylinders are the only tubes in E^3 satisfying the condition $\Delta H = AH$ for some 3×3 matrix A.*

Proof. Let $\sigma : (a, b) \to E^3$ be a smooth unit speed curve of finite length which is topologically imbedded in E^3. The total space N_σ of the normal bundle of $\sigma((a, b))$ in E^3 is naturally diffeomorphic to the direct product $(a, b) \times E^2$ via the translation along σ with respect to the induced normal connection. For a sufficiently small $r > 0$, the tube of radius r about the curve σ is the set:

$$T_r(\sigma) = \{\exp_{\sigma(t)} v : v \in N_{\sigma(t)}, \|v\| = r, a < t < b\}.$$

For sufficiently small r, the tube $T_r(\sigma)$ is a smooth surface in E^3. The position vector of the tube $T_r(\sigma)$ can be expressed as

$$X(t, \theta) = \sigma(t) + r\cos\theta N + r\sin\theta B,$$

where T, N, B denote the Frenet frame of the unit speed curve $\sigma = \sigma(t)$.

We denote by k, τ the curvature and the torsion of the curve σ. Then we have

$$X_t = (1 - rk\cos\theta)T - r\tau\sin\theta N + r\tau\cos\theta B = \tau T + r\tau V,$$

$$X_\theta = -r\sin\theta N + r\cos\theta B = rV,$$

where

$$\gamma = 1 - rk(t)\cos\theta, \quad V = -\sin\theta N + \cos\theta B.$$

The Laplacian Δ of the tube $T_r(\sigma)$ is given by (cf. [4])

$$\Delta = -\frac{1}{\gamma^3}\{r\beta\frac{\partial}{\partial t}-[r\tau\beta+\tau'\gamma-\frac{1}{r}(k\gamma^2\sin\theta)]\frac{\partial}{\partial\theta}+\gamma\frac{\partial^2}{\partial t^2} \tag{4.1}$$
$$-2\tau\gamma\frac{\partial^2}{\partial t\partial\theta}+\frac{1}{r^2}(\gamma^3+r^2\gamma\tau^2)\frac{\partial^2}{\partial\theta^2}\},$$

where $\beta = k'(t)\cos\theta + k(t)\tau(t)\sin\theta$.

If $\beta = 0$, then $\tau = 0$ and k is a constant. Thus, σ lies in a plane circle or in a line. If σ lies in a plane circle, the tube is an anchor ring. In this case, a direct computation shows that it does not satisfy the condition $\Delta H = AH$ for any A. If σ lies in a line, the tube is a circular cylinder.

Now, we assume that $\beta \neq 0$. In this case, a direct computation yields

$$\Delta x = -(\frac{k\cos\theta}{\gamma})n + \frac{1}{r}n, \tag{4.2}$$

$$\Delta^2 x = (\frac{3r\beta^2}{\gamma^5})n + \frac{1}{\gamma^4}P_2(\cos\theta,\sin\theta), \tag{4.3}$$

where P_2 is a E^3-valued polynomial of two variables with coefficients given by some functions of t.

We need the following lemma of [4].

Lemma 11. For any intger $k, \ell \geq 1$ we hav

$$\Delta(\frac{\beta^k}{\gamma^\ell}) = -\frac{\ell(\ell+2)r^2\beta^{k+2}}{\gamma^{\ell+4}} + Q_{k,\ell}(\cos\theta,\sin\theta), \tag{4.4}$$

wher $Q_{k,\ell}$ is a polynomial of two variables with functions of t as coefficients.

By applying (4.1)-(4.4) and by induction, we may obtain

$$\Delta^{k+1}x = \left(\frac{(-1)^{k+1}\cdot(4k-1)!}{2^{2k-1}\cdot(2k-1)!}\right)\left(\frac{r^{2k-1}\beta^{2k}}{\gamma^{4k+1}}\right)n + \gamma^{-4k}P_{k+1}(\cos\theta,\sin\theta), \tag{4.4}$$

for $k \geq 1$, where P_{k+1} is a E^3-valued polynomial of two variables with some functions of t as coefficients.

If the tube statisfies the condition $\Delta H = AH$, then, by Lemma 1, there exists constants c_1, c_2, c_3 such that

$$\Delta^4 x + c_1\Delta^3 x + c_2\Delta^2 x + c_3\Delta x = 0. \tag{4.5}$$

Thus, there exists a polynomial Q of two variables with functions of t as coefficients such that

$$\frac{(k'\cos\theta + k\tau\sin\theta)^{2k}}{1 - rk\cos\theta} = Q(\cos\theta,\sin\theta).$$

Since r is small, this is impossible unless $k = 0$ which implies that the tube is a circular cylinder.

References

[1] C. Baikoussis and D.E. Blair, "On the Gauss map of rules surfaces," *Glasgow Math. J.*, 34(1992), 355-359.

[2] C. Baikoussis, B.Y. Chen and L. Verstraelen, "Ruled surfaces and tubes with finite type Gauss maps," *Tokyo J. Math.* 16(1993), 341-349.

[3] B.Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific, 1984.

[4] B.Y. Chen, "Surfaces of finite type in Euclidean 3-space," *Bull. Soc. Math. Belg. Seŕ B.*, 39(1987), 243-254.

[5] B.Y. Chen, "Null 2-type surfaces in Euclidean space," *Algebra, Analysis and Geomtry*(1988), 1-18.

[6] B.Y. Chen, "Some open problems and conjectures on submanifolds of finite type," *Soochow J. Math.*, 17(1991), 169-188.

[7] B.Y. Chen, "Some classification theorems for submanifolds in Minkowski space-time," *Archiv Math.*, 62 (1994).

[8] B.Y. Chen, "Submanifolds in de Sitter space-time satisfying $\Delta H = \lambda H$," preprint, 1992.

[9] B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, "Ruled surface of finite type," *Bull. Austral. Math. Soc.*, 42(1990), 447-453.

[10] B.Y. Chen and M. Petrovic, "On spectral decomposition of immersions of finite type," *Bull. Austral. Math. Soc.*, 44(1991), 117-129.

[11] F. Dillen, J. Pas and L. Verstraelen, "On surfaces of finite type in Euclidean 3-space," *Kodai Math. J.*, 13(1990), 10-21.

[12] F. Dillen, J. Pas and L. Verstraelen, "On the Gauss map of surfaces of revolution," *Bull. Inst. Math. Acad. Sinica*, 18(1990), 239-246.

[13] O. Garay, "An extension of Takahashi's theorem," *Geom. Dedicata*, 34(1930), 105-112.

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