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ALGEBRAIC EQUIVALENCE OF QUASINORMAL OPERATORS

KUNG-YEU CHEN

Abstract. Let $T_j = N_j \oplus (S \otimes A_j)$ be quasinormal, where N_j is normal and A_j is a positive definite operator, $j = 1, 2$. We show that T_1 is algebraically equivalent to T_2 if and only if $\sigma(A_1) = \sigma(A_2)$ and $\sigma(N_1)\backslash\sigma_{ap}(S\otimes A_1) = \sigma(N_2)\backslash\sigma_{ap}(S\otimes A_2)$. This generalizes the corresponding result for normal and isometric operators.

1. Introduction

For any operator T, let $C^*(T)$ be the C^*-algebra generated by T and I. In other words, $C^*(T)$is the norm-closure of operators of the from $p(T, T^*)$, where p is any polynomial in two noncommuting variables. Two operators T_1 and T_2 are *algebraically equivalent* if there is a *-isomorphism between $C^*(T_1)$ and $C^*(T_2)$ that sends T_1 to T_2. It is easy to prove that two normal operators T_1 and T_2 are algebraically equivalent if and only if they have the same spectrum (Proposition 2.1). It was shown by L. A. Coburn [3, Theorem 3] that each nonunitary isometry is algebraically equivalent to the simple unilateral shift S. The condition on the algebraic equivalence of isometries follows easily (Proposition 2.2). These suggest the following question: when are two quasinormal operators algebrically equivalent? In this paper, we completely solve this problem by using the structure of quasinormal operators (Theorem 2.5).

Before starting our work, we give some notation and terminology. We write D for the open unit disc in the complex plane $\mathbb{C}$ and ∂D for the boundary of D. If H is a Hilbert space, we denote by $\mathcal{B}(H)$ the space of all (bounded linear) operators on H. If T is an operator on H, $\sigma(T)$ and $\sigma_{ap}(T)$ denote its *spectrum* and *approximate point spectrum*. If H is infinite-dimensional, let $\pi(T)$ denote the imageof T in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ under the natural quotient map π, where $\mathcal{K}(H)$ is the ideal of compact operators in $\mathcal{B}(H)$. A bounded linear operator T on a complex separable Hilbert space is *normal* if T and T^* commute; T is *quasinormal* if T and T^*T commute. The

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structure of normal operators has been known for a long time. They are multiplication operators on $L^2(\mu)$ for some positive measure μ. Quasinormal operators were first studied by A. Brown [1] in 1953. Among other things, he obtained a structure theorem for such operators: T is quasinormal if and only if it is unitarily equivalent to an operator of the form $N \oplus (S \otimes A)$, where N is normal and A is a positive definite operator (denoted $A > 0$), that is, $(Ax, x) > 0$ for any nonzero vector x in the domain of A, and

$$S \otimes A = \begin{bmatrix} 0 & & & & & & \\ A & 0 & & & & & \\ 0 & A & 0 & & & & \\ 0 & 0 & A & 0 & & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & & \cdot \;\cdot \end{bmatrix}.$$

Moreover, in such a decomposition, N and A are uniquely determined by T. N and $S \otimes A$ are called the *normal* and *pure parts* of T, respectively ($S \otimes A$ is *pure* means that it has no nontrivial reducing subspace on which it is normal).

Finally, we remark that contents of this paper has formed part of the author's Ph. D. thesis [2].

2. Main Results

We begin with the following proposition. It is a consequence of the Gelfand theory.

Proposition 2.1. *If N_1 and N_2 are two normal operators, then N_1 and N_2 are algebraically equivalent if and only if $\sigma(N_1) = \sigma(N_2)$.*

Proof. One direction is trivial: if α is a *-isomorphism from $C^*(N_1)$ noto $C^*(N_2)$, then $\sigma(T) = \sigma(\alpha(T))$ for any T in $C^*(N_1)$. To prove the converse, observe that, for a normal operator N, $C^*(N)$ is isometrically *-isomorphic to $C(\sigma(N))$ (the C^*-algebra of all continuous functions on $\sigma(N)$) under the mapping $N \leftrightarrow \psi$, where $\psi(z) = z$ for all z in $\sigma(N)$. The assertion follows from this observation immediately.

The following proposition follows easily from [3, Theorem 3].

Proposition 2.2. *If T_1 and T_2 are isometries, then they are algebraically equivalent if and only if either they are both unitary with equal spectra or they are both nonunitary.*

Proposition 2.3. *Let p be a polynomial in two noncommuting variables and $A_j > 0$, $j = 1, 2$. If $\sigma(A_1) = \sigma(A_2)$, then $\|p(S \otimes A_1, (S \otimes A_1)^*)\| = \|p(S \otimes A_2, (S \otimes A_2)^*)\|$.*

Proof. Assume that A_2 is diagonal with eigenvalues $\{a_n\}$. The general case follows from it immediately. In this case, $S \otimes A_2 \cong \sum_n \oplus a_n S$. A straightforward computation shows that

$$p(S \otimes A_2, (S \otimes A_2)^*) \cong \sum_n \oplus p(a_n S, a_n S^*),$$

and hence

$$\|p(S \otimes A_2, (S \otimes A_2)^*)\| = \sup_n \|p(a_n S, a_n S^*)\|.$$

For $\epsilon > 0$, choose $\delta > 0$ such that

$$\|p(X, X^*) - p(Y, Y^*)\| < \epsilon \text{ whenever } \|X\|, \|Y\| \leq \|A_1\| \text{ and } \|X - Y\| \leq \delta. \tag{2.3.1}$$

Assune that A_1 acts on Hilbert space H. Let E_{A_1} be the spectral measure for A_1,

$$B_n = A_1 | E_{A_1}((a_n - \delta, a_n + \delta))H \text{ and } D_n = a_n E_{A_1}((a_n - \delta, a_n + \delta)).$$

Since

$$\|S \otimes B_n - S \otimes D_n\| = \|B_n - D_n\| \leq \delta,$$

it follows from (2.3.1) that

$$\begin{aligned}\|p(S \otimes A_1, (S \otimes A_1)^*)\| &\geq \|p(S \otimes B_n, (S \otimes B_n)^*)\| \\ &\geq \|p(S \otimes D_n, (S \otimes D_n)^*)\| - \epsilon \\ &= \|p(a_n S, a_n S^*)\| - \epsilon\end{aligned}$$

for each n, and hence

$$\begin{aligned}\|p(S \otimes A_1, (S \otimes A_1)^*)\| &\geq \sup_n \|p(a_n S, a_n S^*)\| - \epsilon \\ &= \|p(S \otimes A_2, (S \otimes A_2)^*)\| - \epsilon.\end{aligned} \tag{2.3.2}$$

On the other hand, there exist $\{b_j\}_{j=1}^m$ with $0 < b_j - b_{j-1} \leq \delta$, $b_0 = 0$ and $\sigma(A_1) \subset [0, b_m]$. Let

$$A_{1j} = A_1 | E_{A_1}([b_{j-1}, b_j))H, \qquad 1 \leq j < m$$

and

$$A_{1m} = A_1 | E_{A_1}([b_{m-1}, b_m])H.$$

Then

$$A_1 = \sum_{j=1}^m \oplus A_{1j}.$$

For each j, choose N_j such that

$$\|A_{1j} - a_{n_j}\| < \delta.$$

Again, from (2.3.1), we have

$$\|p(S\otimes A_2,(S\otimes A_2)^*)\| \geq \|p(a_{n_j}S,a_{n_j}S^*)\| \geq \|p(S\otimes A_{1j},(S\otimes A_{1j})^*)\| - \epsilon.$$

Therefore,

$$\|p(S\otimes A_2,(S\otimes A_2)^*)\| \geq \|p(S\otimes A_1,(S\otimes A_1)^*)\| - \epsilon. \tag{2.3.3}$$

Since ϵ is arbitrary, we conclude, by (2.3.2) and (2.3.3), that

$$\|p(S\otimes A_2,(S\otimes A_2)^*)\| = \|p(S\otimes A_1,(S\otimes A_1)^*)\|.$$

This completes the proof.

Lemma 2.4. *If p is a polynomial in two noncommuting variables and $A > 0$, then $\|p(S\otimes A,(S\otimes A)^*)\| \geq \sup\{|p(\lambda,\overline{\lambda})| : |\lambda| \in \sigma(A)\}$.*

Proof. Let B be a diagonal operator with eigenvalues $\{a_n\}$ satisfying $\overline{\{a_n\}} = \sigma(A)$. Since $C^*(S)/\mathcal{K}(H)$ is an abelian C^*-algebra generated by $\pi(S)$, it is $*$-isomorphic to $C(\sigma(\pi(S))) = C(\partial D)$ ([3,Theorem 2]). We have

$$\begin{aligned}\|p(a_nS,a_nS^*)\| \geq& \|p(\pi(a_nS),\pi(a_nS^*))\| \\ =& p(a_n\pi(S),a_n\pi(S^*))\| \\ =& \sup\{|p(a_n\lambda,a_n\overline{\lambda})| : \lambda\in\sigma(\pi(S)) = \partial D\} \\ =& \sup\{|p(\lambda,\overline{\lambda}| : |\lambda| = a_n\}.\end{aligned}$$

Proposition 2.3 leads to that

$$\begin{aligned}\|p(S\otimes A,(S\otimes A)^*)\| =& \|p(S\otimes B,(S\otimes B)^*)\| \\ =& \sup_n \|p(a_nS,a_nS^*)\| \\ \geq& \sup\{|p(\lambda,\overline{\lambda})| : |\lambda| \in \{a_n\}\} \\ =& \sup\{|p(\lambda,\overline{\lambda})| : |\lambda| \in \sigma(A)\}\end{aligned}$$

completing the proof.

The next theorem on algebraic equivalence for quasinormal operators generalizes Propositions 2.1 and 2.2.

Theorem 2.5. *Let $T_j = N_j \oplus (S\otimes A_j)$ be quasinormal, where N_j is normal and $A_j > 0$, $j = 1,2$. Then T_1 is algebraically equivalent to T_2 if and only if $\sigma(A_1) = \sigma(A_2)$ and $\sigma(N_1)\backslash\sigma_{ap}(S\otimes A_1) = \sigma(N_2)\backslash\sigma_{ap}(S\otimes A_2)$.*

Proof. Let α be a $*$-isomorphism from $C^*(T_1)$ noto $C^*(T_2)$ such that $\alpha(T_1) = T_2$. We have $\alpha(T_1^*T_1 - T_1T_1^*) = T_2^*T_2 - T_2T_2^*$, that is $\alpha(A_1^2\oplus 0) = A_2^2\oplus 0$. Consequently, $\sigma(A_1^2\oplus 0) = \sigma(A_2^2\oplus 0)$. Therefore, $\sigma(A_1^2)\cup\{0\} = \sigma(A_2^2)\cup\{0\}$. Since 0 cannot be an isolated point in the spectrum of a positive definite operator, we conclude that $\sigma(A_1^2) = \sigma(A_2^2)$, and hence

$$\sigma(A_1) = \sigma(A_2). \tag{2.5.1}$$

Next, observe that, for an operator T, $\lambda \notin \sigma_{ap}(T)$ is equivalent to that

$$(T-\lambda)^*(T-\lambda) \geq \delta \text{ for some } \delta > 0.$$

This, together with order-preserving property of α, shows that

$$\sigma_{ap}(T_1) = \sigma_{ap}(T_2). \tag{2.5.2}$$

Since

$$\begin{aligned}\sigma(N_j)\backslash\sigma_{ap}(S\otimes A_j) =&(\sigma(N_j)\cup\sigma_{ap}(S\otimes A_j))\backslash\sigma_{ap}(S\otimes A_j)\\ =&\sigma_{ap}(T_j)\backslash\sigma_{ap}(S\otimes A_j),\ j=1,2,\end{aligned}$$

it follows from (2.5.1) and (2.5.2) that

$$\sigma(N_1)\backslash\sigma_{ap}(S\otimes A_1) = \sigma(N_2)\backslash\sigma_{ap}(S\otimes A_2).$$

This proves the necessity condition.

To prove the sufficiency, it suffices to show that $\|p(T_1,T_1^*)\| = \|p(T_2,T_2^*)\|$ for any polynomial p. To do this, observe that

$$\begin{aligned}&\|p(T_j,T_j^*)\|\\ =&\|p(N_j,N_j^*)\oplus p(S\otimes A_j,(S\otimes A_j)^*)\|\\ =&\max\{\|p(N_j,N_j^*)\|,\|p(S\otimes A_j,(S\otimes A_j)^*)\|\}\\ =&\max\{\sup\{|p(\lambda,\overline{\lambda})|:\lambda\in\sigma(N_j)\},\|p(S\otimes A_j,(S\otimes A_j)^*)\|\}\\ =&\max\{\sup\{|p(\lambda,\overline{\lambda}):\lambda\in\sigma(N_j)\backslash\sigma_{ap}(S\otimes A_j)\},\\ &\sup\{|p(\lambda,\overline{\lambda})|:\lambda\in\sigma(N_j)\cap\sigma_{ap}(S\otimes A_j)\},\|p(S\otimes A_j,(S\otimes A_j)^*)\|\}\\ =&\max\{\sup\{|p(\lambda,\overline{\lambda})|:\lambda\in\sigma(N_j)\backslash\sigma_{ap}(S\otimes A_j)\},\|p(S\otimes A_j,(S\otimes A_j)^*)\|\}\end{aligned}$$

by Lemma 2.4 and $\sigma_{ap}(S\otimes A_j) = \{Z : |z| \in \sigma(A_j)\}$ ([2, Lemma 2.2]). Our assumption and Proposition 2.3 imply that $\|p(T_1,T_1^*)\| = \|p(T_2,T_2^*)\|$. This completes the proof.

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Departement of Mathematics, Tamkang University, Tamsui, Taiwan 25137, Republic of China.