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ON (N, p_n, q_n) SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

NARENDRA KUMAR SHARMA AND RAJIV SINHA

Abstract. The aim of the present paper is to generalize the result of the theorems given by Pandey [4].

1. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{S_n\}$. Let p denote the sequence $\{p_n\}$, $p_{-1} = 0$, given two sequences p and q the convolution $(p * q)$ is defined by

$$(p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k$$

for any sequence $\{S_n\}$ we write

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_{n-k} q_k S_k \tag{1.1}$$

If $(p * q)_n \neq 0$ for all n. If $t_n^{p,q} \to S$ as $n \to \infty$, we write

$$\sum_{n=0}^{\infty} a_n = S(N, p_n, q_n) \quad \text{or} \quad S \to S(N, p_n, q_n)$$

The necessary and sufficient condition for $S(N, p_n, q_n)$ method to be regular are $\sum_{k=0}^{n} |p_{n-k} q_k| = O(|(p * q)_n|)$ and $p_{n-k} = O(|p * q)_n|)$ as $n \to \infty$, for fixed $k \geq 0$ for each $q_k \neq 0$.

2. Let $f(t)$ be a 2π-periodic function and Lebesgue integrable over an interval $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t) \tag{2.1}$$

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And then conjugate series of (2.1) is

$$\sum_{n=1}^{\infty}(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} B_n(t) \tag{2.2}$$

we shall use the following notation:

$$\phi(t) = \phi(x,t) = f(x+t) + f(x-t) - 2f(x)$$

and

$$\Psi(t) = \Psi(x,t) = f(x+t) - f(x-t)$$

$$P\left(\frac{1}{t}\right) = P_{[\frac{1}{t}]}, \quad q\left(\frac{1}{t}\right) = q_{[\frac{1}{t}]}, \quad R\left(\frac{1}{t}\right) = R_{[\frac{1}{t}]},$$

where $\tau = [\frac{1}{t}]$ denotes the integral part of $\frac{1}{t}$

$$R_n = (p * q)_n$$

$$\Phi(t) = \int_0^t |\phi(u)|du, \quad \Psi(t) = \int_0^t |\psi(u)|du$$

$$N_n(t) = \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v} \frac{\sin(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t}$$

$$\overline{N}_n(t) = \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v} \frac{\sin(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t}$$

3. Pandey [4] proved the following two theorems.

Theorem 1. *If* $\Phi(t) = \int_0^t |\phi(u)|du = O[tX(t)]$ (3.1)
as $t \to +0$, *where* $X(t)$ *is a positive, non-decreasing function of* t, *such that*

$$X(\frac{1}{n}) = O(1) \qquad as \quad n \to \infty \tag{3.2}$$

and

$$\int_{\frac{1}{n}}^{\delta} X(t)Q_\tau \frac{dt}{t} = O(Q_n), \qquad as \quad n \to \infty \tag{3.3}$$

Then the Fourier series of $f(t)$ *at* $t = x$ *is summable* (N, q_n) *to* $f(x)$, *where* $\{q_n\}$ *is a real, non-negative and non-increasing sequence such that* $Q_n \to \infty$ *as* $n \to \infty$.

Theorem 2. *If the sequence* $\{q_n\}$ *and* $\{X(t)\}$ *be same as in Theorem 1. Then if*

$$\Psi(t) \equiv \int_0^t |\psi(t)|du = O[tX(t)] \tag{3.4}$$

as $t \to +0$. Then the conjugate series (2.2) is summable (N, q_n) to

$$\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{t}{2}dt$$

at every point where its integral exists.

4. The object of this paper is to obtain, more general theorems for (N, p_n, q_n) Nörlund summability of the Fourier series and its conjugate series.

We shall prove the following theorems.

Theorem A. *If $\Phi(t) = \int_0^t |\phi(u)|du = O[tX(t)]$* (4.1)
as $t \to +0$, where $X(t)$ is a positive non-decreasing function of t, such that

$$X(\frac{1}{n}) = O(1) \qquad as \quad n \to \infty \tag{4.2}$$

and

$$\int_{\frac{1}{n}}^{\delta} X(t)R_\tau \frac{dt}{t} = O(R_n) \qquad as \quad n \to \infty \tag{4.3}$$

Then the Fourier series at $t = x$ is summable (N, p_n, q_n). Where $\{p_n\}$ and $\{q_n\}$ are real, non-negative and non-increasing sequence such that $R_n \to \infty$, as $n \to \infty$.

Theorem B. *If the sequences $\{p_n\}$, $\{q_n\}$ and $\{X(t)\}$ be same as in Theorem [A]. then if*

$$\Psi(t) \equiv \int_0^t |\psi(u)|du = O(tX(t)) \tag{4.4}$$

as $t \to +0$. Then the conjugate series (2.2) is summable (N, p_n, q_n) to

$$\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{t}{2}dt$$

at every point where its integral exists.

5. We shall use the following lemmas in the proof of our theorems.

Lemma 1.[3]. *If $\{p_n\}$ and $\{q_n\}$ are non-negative non-increasing sequences, then for $0 \le a \le b \le \infty$, $0 < t < \pi$.*

We have

$$|\sum_{k=a}^{b} p_k q_{n-k} e^{i(n-k)t}| \le R(\frac{1}{t}) \qquad \text{for any} \quad a.$$

Lemma 2.[5,6] *For the sequences $\{p_n\}$ and $\{q_n\}$ satisfying the conditions of* *theorem,* $0 < t < \pi$

$$N_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{t}{2}} = O\left[\frac{R(\frac{1}{t})}{tR_n}\right]$$

and

$$\overline{N}_n(t) = \frac{1}{2\pi R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{\cos(n-k+\frac{1}{2})t}{\sin\frac{t}{2}} = O\left[\frac{R(\frac{1}{t})}{tR_n}\right]$$

and for $0 \le t \le \frac{1}{n}$ $N_n(t) = \overline{N}_n(t) = O(n)$

Proof of the Theorem 1.

Let

$$S_n(x) = \sum_{v=1}^{n} A_v(x) + \frac{1}{2}a_0$$

Then we write-

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

using (1.1) we get

$$\begin{aligned} t_n^{p,q} - f(x) =& \frac{1}{R_n}\sum_{v=0}^{n} p_v q_{n-v}[S_{n-v} - f(x)] \\ =& \frac{1}{R_n}\sum_{v=0}^{n} p_v q_{n-v}\frac{1}{2\pi}\phi(t)\frac{\sin(n-v+\frac{1}{2})t}{\sin\frac{t}{2}}dt \\ =& \int_0^{\pi}\phi(t)\left\{\frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v}\frac{\sin(n-v+\frac{1}{2})t}{\sin\frac{t}{2}}dt\right\} \\ =& \int_0^{\pi}\phi(t)N_n(t)dt, \qquad \text{say.} \end{aligned}$$

In order to prove the theorem, we have to show that

$$\int_0^{\pi}\phi(t)N_n(t)dt = O(1), \qquad \text{as} \quad n\to\infty$$

we write for $0 < \delta < \pi$

$$\begin{aligned} \int_0^{\pi}\phi(t)N_n(t)dt =& \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi}\right]\phi(t)N_n(t)dt \\ =& I_1 + I_2 + I_3, \qquad \text{say} \end{aligned}$$

Now

$$I_1 = \int_0^{\frac{1}{n}} \phi(t)N_n(t)dt$$
$$= O\left[n\int_0^{\frac{1}{n}} |\phi(t)|dt\right] = O(1)$$

$$I_2 = \int_{\frac{1}{n}}^{\delta} \phi(t)N_n(t)dt$$
$$= O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} |\phi(t)|\frac{R_\tau}{t}dt]$$
$$= O[R_n^{-1}\frac{\Phi(t)R_\tau}{t}]_{\frac{1}{n}}^{\delta} + O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} \Phi(t)\frac{R_\tau}{t^2}dt]$$
$$+ O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} \Phi(t)\frac{1}{t}dR_\tau]$$
$$= O[R_n^{-1}\{tX(t)\frac{R_\tau}{t}\}_{\frac{1}{n}}^{\delta}] + O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} tX(t)\frac{R_\tau}{t^2}dt]$$
$$+ O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} tX(t)\frac{1}{t}|dR_\tau|]$$

Now by the hypothesis (4.1) and (4.3) of the theorem

$$= O(1) + O(1) + O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} X(t)\, d_{R[\tau]}]$$
$$= O(1) + O(1) + O[R_n^{-1}\int_{\frac{1}{n}}^{\delta} X(t)\, dR_{[\tau]}]$$
$$= O(1) + O(1) + O[R_n^{-1}\sum_{v=0}^{n} R_v]$$
$$= O(1) + O(1) + O(1)$$
$$= O(1) \qquad \text{as} \quad n \to \infty.$$

Lastly, by virtue of Riemann Lebesgue theorem and regularity of the method of summation, we have

$$I_3 = \int_{\delta}^{\pi} \phi(t)N_n(t)\, dt = O(1) \qquad \text{as} \quad n \to \infty$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let $\overline{S}_n(x)$ denote the nth partial sum of the series $\sum B_n(x)$. Then we have

$$\overline{S}_n(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(t)\frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

For $\sum B_n(x)$, making use of the formula (1.1)

$$t_n^{p,q} - \frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{t}{2}\,dt$$
$$=R_n^{-1}\sum_{v=0}^{n} p_v q_{n-v}\overline{s}_{n-v}(x) - \frac{1}{2}\int_0^\pi \psi(t)\cot\frac{t}{2}dt$$
$$=R_n^{-1}\sum_{v=0}^{n} p_v q_{n-v}\frac{1}{2\pi}\int_0^\pi \Psi(t)\frac{\cos\frac{t}{2} - \cos(n-v+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$
$$-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}dt$$
$$=-\int_0^\pi \psi(t)\{\frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v}\frac{\cos(n-v+\frac{1}{2})t}{\sin\frac{t}{2}}\}dt$$
$$=-\int_0^\pi \psi(t)\overline{N}_n(t)\,dt. \quad (\text{say}, = H).$$

In order to prove the theorem, we have to show that, under our assumptions

$$\int_0^\pi \psi(t)\overline{N}_n(t)dt = O(1) \qquad \text{as} \quad n\to\infty$$

for $0<\delta<\pi$, we have

$$\int_0^\pi \psi(t)\overline{N}_n(t)dt = \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int^{\pi}\right]\psi(t)\overline{N}_n(t)\,dt$$
$$=H_1 + H_2 + H_3, \qquad \text{say},$$

since the conjugate function exists, Therefore

$$\frac{1}{2\pi}\int_0^{\frac{1}{n}} \psi(t)\cot\frac{t}{2}dt = O(1)$$

and

$$\frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v}\frac{\cos\frac{t}{2} - \cos(n-v+\frac{1}{2})t}{\sin\frac{t}{2}}$$
$$=\frac{1}{2\pi R_n}\sum_{v=0}^{n} p_v q_{n-v}\sum_{k=0}^{n} 2\sin kt$$
$$=O[R_n^{-1}\sum_{v=0}^{n} p_v q_{n-v}\sum_{k=0}^{n-v}|\sin kt|]$$
$$=O[R_n^{-1}\sum_{v=0}^{n} p_v q_{n-v}(n-v)]$$
$$=O(n), \qquad \text{for } 0\le t\le\pi$$

Therefore,

$$
\begin{aligned}
H_1 &= \int_0^{\frac{1}{n}} \psi(t)\overline{N}_n(t)\,dt \\
&= \int_0^{\frac{1}{n}} \frac{\psi(t)}{2\pi R_n} \sum_{v=0}^{n} p_v q_{n-v} \frac{\cos(n-v+\frac{1}{2})t}{\sin\frac{t}{2}} dt \\
&= -\int_0^{\frac{1}{n}} \frac{\psi(t)}{2\pi R_n} \sum_{v=0}^{n} p_v q_{n-v} \frac{\cos\frac{t}{2} - \cos(n-v+\frac{1}{2})t}{\sin\frac{t}{2}} dt \\
&\quad + \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_v q_{n-v} \int_0^{\frac{1}{n}} \psi(t)\cot\frac{t}{2}\,dt \\
&= O(n\int_0^{\frac{1}{n}} |\psi(t)|dt) + O(1) \\
&= O[n\Psi(\frac{1}{n})] + O(1) \\
&= O(1) + O(1) = O(1) \qquad \text{as } n \to \infty
\end{aligned}
$$

Now for $\frac{1}{n} \le t \le \delta$

$$
\begin{aligned}
H_2 &= O[\int_{\frac{1}{n}}^{\delta} |\psi(t)|\ |\overline{N}_n(t)|dt] \\
&= O[\int_{\frac{1}{n}}^{\delta} |\psi(t)|\ \frac{R_\tau}{R_n t}] \\
H_2 &= O(1) \qquad \text{as in } I_2
\end{aligned}
$$

also $H_3 = O(1)$

By virtue of the Riemann Lebesgue theorem and the regularity of the method of summation, on calculating H_1, H_2, H_3 we get

$$H = O(1)$$

which completes the proof of the Theorem 2.

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Department of Mathematics, S.M. Post Graduate College, Chandausi-202412, India.