TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1994

CRITERIA FOR DICHOTOMY OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In the present paper necessary and sufficient conditions for (μ_1, μ_2)-dichotomy of linear impulsive differential equations are obtained without imposing conditions of bounded growth on these equations. The apparatus of piecewise continuous Lyapunov's functions is used.

1. Introduction

Let $\mathbf{Z}$ be the set of all integers. S be the set of real or complex numbers, and let $J = (\omega_-, \omega_+) \subset \mathbf{R}$ be a real interval which can be bounded or unbounded. Consider the linear impulsive differential equations

$$\begin{aligned} x' &= A(t), \qquad t \neq \tau_k, \\ x^+ &= A_k x, \qquad t = \tau_k, \end{aligned} \tag{1}$$

where $x \in S^n$, $t \in J$, $k \in \mathbf{Z}$, $A_k \in S^{n\times n}$ is an $n \times n$ -matrix with entries of S and the moments τ_k of impulse effect satisfy the conditions

$$\lim_{k\to\pm\infty} \tau_k = \omega_\pm, \quad \tau_k < \tau_{k+1} \quad (k \in \mathbf{Z}).$$

Denote by $PC(J, S^{n\times m})$ the space of functions $f : J \to S^{n\times m}$ which are continuous for $t \neq \tau_k$ and for $t = \tau_k$ they have discontinuities of the first kind and are continuous from the left. We shall recall [1] that by a *solution* of (1) we mean any function $x : J \to S^n$

Received September 30, 1992.

1991 *Mathematics Subject Classification.* 34A37.

Key words and phrases. Dichotomy, impulsive differential equations.

This investigation was supported by the Bulgarian Ministry of Science and Higher Education under Grant MM-7.

which is differentiable for $t \neq \tau_k$ and satisfies the equation $x' = A(t)x$ and for $t = \tau_k$ satisfies the conditions

$$x(\tau_k^-) \stackrel{\text{def}}{=} \lim_{t \to \tau_k - 0} x(t) = x(\tau_k), \quad x(\tau_k^+) \stackrel{\text{def}}{=} \lim_{t \to \tau_k + 0} x(t) = A_k x(\tau_k).$$

Assume the following conditions fulfilled:
A1. $A(t) \in PC(J, S^{n \times n})$.
A2. $\det A_k \neq 0$ $(k \in \mathbf{Z})$.

Under this assumption, all solutions $x(t)$ of (1) are defined in J and form an n-dimensional space of solutions which we denote by X. Let $|\cdot|$ denote some norm in S^n and also the corresponding matrix norm. Let $X(t)$ be a fundamental matrix of solutions of equation (1) and let the functions $\mu_1, \mu_2 \in PC(J, \mathbf{R})$.

Definition 1. Equation (1) is said to have a (μ_1, μ_2)-dichotomy if there exist supplementary projections P_1, P_2 on S^n such that

$$|X(t)P_iX^{-1}(s)| \leq K_i \exp\left(\int_s^t \mu_i(\tau)d\tau\right) \quad (-1)^i(s-t) \geq 0, \ i = 1, 2,$$

where $K_1, K_2 \geq 1$ are constants.

In the case when μ_1, μ_2 are constants equation (1) is said to have an *exponential dichotomy* if $\mu_1 < 0 < \mu_2$ and *ordinary dichotomy* if $\mu_1 = \mu_2 = 0$.

Condition (2) is equivalent to the conditions

$$|X(t)P_i\xi| \leq L_i \exp\left(\int_s^t \mu_i(\tau)d\tau\right)|X(s)P_i\xi| \quad \text{if } (-1)^i(s-t) \geq 0, \ i = 1, 2, \tag{3}$$

$$|X(t)P_iX^{-1}(t)| \leq M_i \tag{4}$$

for any vector $\xi \in S^n$, where L_i, $M_i \geq 1$ are constants.

If the projector P_i has rank k_i, $i = 1, 2$, $k_1 + k_2 = n$, then condition (3) means that the space of solutions X has two supplementary subspaces X_1, X_2 of dimensions k_1, k_2 such that

$$|x(t)| \leq L_1 \exp\left(\int_s^t \mu_1(\tau)d\tau\right)|x(s)| \qquad (t \geq s, \quad x \in X_1)$$

$$|x(t)| \leq L_2 \exp\left(\int_s^t \mu_2(\tau)d\tau\right)|x(s)| \qquad (s \geq t, \quad x \in X_2)$$

Condition (4) means that the supplementary projectors $X(t)P_iX^{-1}(t)$ from S^n onto the subspaces $S_i(t) = \{x(t) \in S^n : x \in X_i\}$, $i = 1, 2$ are bounded uniformly on $t \in J$, or equivalently, that the "angle" between the subspaces $S_i(t)$, $i = 1, 2$ is bounded away from zero for $t \in J$ (of. [2], p.156).

Some criteria for exponential dichotomy are well known [3]. However, the sufficient conditions usually require equation (1) to have a bounded growth (of. [3], Lectures 1,6,8).

In the present paper three necessary and sufficient conditions for (μ_1, μ_2)-dichotomy without such constraints on the growth are given.

The proofs of the theorems are close to those by J.S. Muldowney of [4]. As an apparatus piecewise continuous comparison functions are used, which were introduced in [5] for investigation of the stability of the solutions of the impulsive differential equations by Lyapunov's direct method.

2. Preliminary notes.

We shall give some definitions and notation to be used henceforth.

Definition 2[5]. The function $U : J \times S^n \to \mathbf{R}$. $(t, x) \to U(t, x)$ is said to belong to the class V_0 if:

1. U is continuous and locally Lipschitz continuous with respect to x in the domain $G_k = (\tau_k, \tau_{k+1}) \times S^n$ $(k \in \mathbf{Z})$.

2. For any $k \in \mathbf{Z}$ and $x \in S^n$ there exist the finite limits

$$U(\tau_k^-, x) = \lim_{\substack{(t,y)\to(\tau_k,x)\\(t,y)\in G_{k-1}}} U(t, y), \qquad U(\tau_k^+, x) = \lim_{\substack{(t,y)\to(\tau_k,x)\\(t,y)\in G_k}} U(t, y)$$

and $U(\tau_k^-, x) = U(\tau_k, x)$.

For the function $U \in V_0$ and $t \neq \tau_k$, $x \in S^n$ define

$$\dot{U}(t, x) = \limsup_{h\to 0_+} \frac{1}{h}[U(t + h, x + hA(t)x) - U(t, x)]$$

- upper right derivative of the function U with respect to equation (1).

We shall recall [6] that if $x(t)$ is a solution of (1), $U \in V_0$ and $u(t) = U(t, x(t))$, then

$$D^+u(t) = \dot{U}(t, x(t)) \quad (t \neq \tau_k),$$

where D^+u is the upper right Dini derivative of the function u.

Definition 3. The couple of functions $V_i(t, x) \in V_0$, $i = 1, 2$ is said to be *admissible* if for any $t \in J$ there exist supplementary projectors $Q_1(t), Q_2(t)$ of rank k_1, k_2 independent of t such that

$$|Q_i(t)| \leq N_i \quad (i = 1, 2), \tag{5}$$

$$|Q_i(t)x|^r \leq V_i(t, x) \leq b_i|Q_i(t)x|^r \quad (i = 1, 2) \tag{6}$$

for any $(t, x) \in J \times S^n$, where $N_i, b_i, r > 0$ are constants.

When the admissible couple is given, i.e. the couple of projectors Q_i $(i = 1, 2)$ and the number r are determined uniquely, we shall always choose for N_i, b_i the least possible values for which (5) and (6) are satisfied.

If $V_1(t,x)$ and $V_2(t,x)$ is an admissible couple and $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_i \geq 0$, then we define

$$V(\lambda; t, x) = \lambda_1 V_1(t,x) - \lambda_2 V_2(t,x).$$

3. Main results

Theorem 1. *Let condition (A) hold and let there exist an admissible couple $V_1(t,x)$, $V_2(t,x)$ and real numbers ℓ_1, ℓ_2 such that $0 \leq \ell_i b_i < 1$, $i = 1,2$ and*

$$\dot V(\lambda; t,x) \leq \rho_\lambda(t) V(\lambda; t,x) \quad (if\ V(\lambda;t,x) \geq 0,\ t \neq \tau_k), \tag{7}$$

$$\dot V(\lambda; t,x) \leq \delta_\lambda(t) V(\lambda; t,x) \quad (if\ V(\lambda;t,x) \leq 0,\ t \neq \tau_k), \tag{8}$$

$$V(\lambda; \tau_k^+, A_k x) \leq V(\lambda; \tau_k, x) \qquad (k \in \mathbf{Z}) \tag{9}$$

for $\lambda = (1, \ell_2)$ and $\lambda = (\ell_1, 1)$, where $\rho_\lambda, \delta_\lambda \in PC(J, \mathbf{R})$ and $\rho_\lambda = r\mu_1$ if $\lambda = (1,\ell_2)$, $\delta_\lambda = r\mu_2$ if $\lambda = (\ell_1, 1)$.

Then equation (1) has a (μ_1, μ_2)-dichotomy.

Theorem 2. *Let conditions (A) hold and let a function $\rho \in PC(J, \mathbf{R})$ exist such that $\mu_1 \leq \rho \leq \mu_2$, as well as an admissible couple $V_1(t,x)$, $V_2(t,x)$ and real numbers ℓ_1, ℓ_2, $0 < \ell_i b_i < 1$, $i = 1,2$ such that*

$$\dot V_1(t,x) \leq r\rho(t) V_1(t,x) \quad (if\ V_1(t,x) \geq \ell_2 V_2(t,x),\ t \neq \tau_k), \tag{10}$$

$$\dot V_2(t,x) \geq r\mu_2(t) V_2(t,x) \quad (if\ V_1(t,x) \leq \ell_2 V_2(t,x),\ t \neq \tau_k), \tag{11}$$

$$\dot V_1(t,x) \leq r\mu_1(t) V_1(t,x) \quad (if\ \ell_1 V_1(t,x) \geq V_2(t,x),\ t \neq \tau_k), \tag{12}$$

$$\dot V_2(t,x) \geq r\rho(t) V_2(t,x) \quad (if\ \ell_1 V_1(t,x) \leq V_2(t,x),\ t \neq \tau_k), \tag{13}$$

$$V_1(\tau_k^+, A_k x) \leq V_1(\tau_k, x) \qquad (k \in \mathbf{Z}), \tag{14}$$

$$V_2(\tau_k^+, A_k x) \geq V_2(\tau_k, x) \qquad (k \in \mathbf{Z}). \tag{15}$$

Then equation (1) has a (μ_1, μ_2)-dichotomy.

Theorem 3. *Let conditions (A) hold and let equation (1) have a (μ_1, μ_2)-dichotomy. Then there exists an admissible couple $V_1(t,x)$, $V_2(t,x)$ such that*

$$\dot V_1(t,x) \leq r\mu_1(t) V_1(t,x) \qquad (t \neq \tau_k), \tag{16}$$

$$\dot V_2(t,x) \geq r\mu_2(t) V_2(t,x) \qquad (t \neq \tau_k), \tag{17}$$

$$V_1(\tau_k^+, A_k x) \leq V_1(\tau_k, x) \qquad (k \in \mathbf{Z}), \tag{18}$$

$$V_2(\tau_k^+, A_k x) \geq V_2(\tau_k, x) \qquad (k \in \mathbf{Z}), \tag{19}$$

Corollary 1. *Let conditions (A) hold. Then:*

(a) The condition given as sufficient for a (μ_1, μ_2)-dichotomy in Theorem 1, are also necessary.

(b) When $\mu_1 \le \mu_2$ the condition given as sufficient for a (μ_1, μ_2)-dichotomy in Theorem 2, are also necessary.

(c) The condition given as necessary for a (μ_1, μ_2)-dichotomy in Theorem 3, are also sufficient.

Proof of Corollary 1. Assertion (b) is obvious since if the admissible couple $V_1(t,x)$, $V_2(t,x)$ satisfies condition (16)-(21), then it satisfies also the conditions of Theorem 2. Assertions (a) and (c) follow from the fact that the conditions of Theorem 3 imply the conditions of Theorem 1 with $\ell_1 = \ell_2 = 0$. We shall just note that if $U_2(t,x) = -V_2(t,x)$, then condition (17) implies that $\dot{U}_2(t,x) \le r\mu_2(t)U_2(t,x)$ for $(t,x) \in J \times S^n$, $t \ne \tau_k$. The proof of this assertion is carried out as in [4], that is why we omit it.

In the proof of Theorem 1 and Theorem 2 we shall use the following lemma.

Lemma 1 [4]. *Suppose that P_i, $i = 1,2$ and Q_i, $i = 1,2$ are two couples of supplementary projectors in S^n such that*

$$|Q_i| \le N \qquad (i = 1,2),$$

if $\tau < 1$ is a number such that

$$\tau \mid Q_1 P_1 \mid \ge \mid Q_2 P_1 \mid, \qquad \tau |Q_2 P_2| \ge |Q_1 P_2|,$$

then

$$\mid P_i \mid \le 2N \frac{1+\tau}{1-\tau} \qquad (i = 1,2),$$

Proof of Theorem 1. Let $t_0 \in J$ and

$$W(\lambda; t, x) = \begin{cases} \exp(-\int_{t_0}^t \rho_\lambda(\tau)d\tau)V(\lambda; t, x) & \text{if } V(\lambda; t, x) \ge 0 \\ \exp(-\int_{t_0}^t \delta_\lambda(\tau)d\tau)V(\lambda; t, x) & \text{if } V(\lambda; t, x) \le 0 \end{cases}$$

From (7)-(9) it follows that if $x \in X$ then

$$D^+W(\lambda; t, x) \le 0 \qquad (t \ne \tau_k),$$

$$W(\lambda; \tau_k^+, x(\tau_k^+)) \le W(\lambda; \tau_k, x(\tau_k)) \quad (k \in \mathbb{Z}).$$

Therefore, the function $W(\lambda; t, x(t))$ is nonincreasing in J if $x(t)$ is a solution of (1) and $\lambda = (1, \ell_2)$ or $\lambda = (\ell_1, 1)$. In particular, if $\tau \in J$ and $0 \ne x(\tau) \in Q_1(\tau)S^n$, then from (6) $V_1(\tau, x(\tau)) > 0$, $V_2(\tau, x(\tau)) = 0$ since $Q_2(\tau)x(\tau) = 0$. Then

$$W(\lambda; t, x(t)) \ge W(\lambda; \tau, x(\tau)) = \lambda_1 \exp(-\int_{t_0}^{\tau} \rho_\lambda(u)du)V_1(\tau, x(\tau)) > 0 \qquad (t \le \tau).$$

Choose a sequence $\tau_m \in J$, $\tau_m \to \omega_+$. Then for each m there exists a k_1-dimensional subspace of solutions of (1) for which $W(\lambda; t, x(t))$ is nonnegative and nonincreasing in $(\omega_-, \tau_m]$. Let $Y_m(t)$ be an $n \times k_1$ -matrix of solutions of (1) whose columns span this subspace and let the columns of $Y_m(\tau_0)$ be orthonormal. From the compactness of the unit sphere in S^n it follows that a subsequence of $Y_m(\tau_0)$ (without loss of generality the sequence itself) converges to a matrix $Y(\tau_0)$ whose k_1 columns are orthonormal. Thus $\lim_{m\to\infty} Y_m(t) = Y(t)$ for any $t \in J$, where $Y(t)$ is an $n \times k_1$ -matrix of solutions of (1) which has rank k_1. If $\xi \in S^{k_1}$, $x_m(t) = Y_m(t)\xi$ and $x(t) = Y(t)\xi$, then $W(\lambda; t, x_m(t)) \le 0$, $\omega_- < t \le \tau_m$ implies $W(\lambda; t, x(t)) \le 0$, $\omega_- < t < \omega_+$. These conclusion are also valid for $\lambda = (1, \ell_2)$ and for $\lambda = (\ell_1, 1)$. Thus, if x belongs to the k_1-dimensional space

$$X_1 = \{x \in X : x(t) = Y(t)\xi, \quad \xi \in S^{k_1}\}$$

of solution of (1), then

$$\begin{aligned} V_1(t, x(t)) - \ell_2 V_2(t, x(t)) \ge 0 \qquad (t \in J), \\ \ell_1 V_1(t, x(t)) - V_2(t, x(t)) \ge 0 \qquad (t \in J). \end{aligned} \tag{21}$$

Therefore, if $x \in X_1$ and $\lambda = (1, \ell_2)$ or $\lambda = (\ell_1, 1)$, then

$$W(\lambda; t, x(t)) = \exp(-\int_{t_0}^{t} \rho_\lambda(u)du) V(\lambda; t, x(t))$$

and this function is nonincreasing in J. In particular, for $\lambda = (1, \ell_2)$

$$V_1(t, x(t)) - \ell_2 V_2(t, x(t)) \le \exp(\int_s^t r\mu_1(u)du)[V_1(s, x(s)) - \ell_2 V_2(s, x(s))] \quad (t \ge s)$$

which together with (21) implies

$$(1 - \ell_1\ell_2) V_1(t, x(t)) \le \exp(\int_s^t r\mu_1(u)du) V_1(s, x(s)) \quad (t \ge s).$$

Since $b_i \ge 1$, then $0 < \ell_i < 1$. Thus $1 - \ell_1\ell_2 > 0$ and from (6)

$$|Q_1(t)x(t)| \le b_1^{1/r}(1 - \ell_1\ell_2)^{-1/r} \exp(\int_s^t \mu_1(u)du) |Q_1(s)x(s)| \quad (t \ge s) \tag{22}$$

From (6) and (21) it follows that

$$(\ell_1 b_1)^{1/r} |Q_1(t)x(t)| \ge |Q_2(t)x(t)| \quad (t \in J, x \in X_1) \tag{23}$$

thus

$$\begin{aligned}|x(t)| =&|Q_1(t)x(t) + Q_2(t)x(t)|\\ \leq&|Q_1(t)x(t)| + |Q_2(t)x(t)|\\ \leq&[1 + (\ell_1 b_1)^{1/r}] \mid Q_1(t)x(t) \mid .\end{aligned}$$

This, together with $|Q_1(s)x(s)| \leq N_1|x(s)|$ (from (5) and (22)) yields

$$|x(t)| \leq L_1 \exp(\int_s^t \mu_1(u)du)|x(s)| \quad (t \geq s, x \in X_1), \tag{24}$$

where $L_1 = b_1^{1/r}(1 - \ell_1\ell_2)^{-1/r}[1 + (\ell_1 b_1)^{1/r}]N_1$.

Similar arguments show that there exists a k_2-dimensional subspace X_2 of solutions of (1) such that

$$(\ell_2 b_2)^{1/r}|Q_2(t)x(t)| \geq |Q_1(t)x(t)| \qquad (t \in J, x \in X_2), \tag{25}$$

$$|x(t)| \leq L_2 \exp(\int_s^t \mu_2(u)du)|x(s)| \qquad (s \geq t, x \in X_2). \tag{26}$$

Since $0 < \ell_i b_i < 1$, then from inequalities (23) and (24) it follows that the spaces X_1, X_2 are supplementary. That is why from (24) and (26) it follows that there exist supplementary projectors P_1, P_2 in S^n such that (4) is valid. Finally, (5), (23) and (25) show that the conditions of Lemma 1 are satisfied for any $t \in J$ for the projectors $Q_i(t)$, $P_i(t) = X(t)P_iX^{-1}(t)$ with $\tau = \max\{(\ell_1 b_1)^{1/r}, (\ell_2 b_2)^{1/r}\}$ and $N = \max\{N_1, N_2\}$. That is why (20) imlies that (4) holds.

Proof of Theorem 2. First we suppose that $\rho = 0$. Let $x(t)$ be an arbitrary solution of (1). Then from (10) and (14) it follows that $V_1(t, x(t))$ is nonincreasing in the interval $I \subset J$ if $V_1(t, x(t)) \geq \ell_2 V_2(t, x(t))$ for any $t \in I$. Similary, from (13) and (15) it follows that $V_2(t, x(t))$ is nondecreasing in I if $\ell_1 V_1(t, x(t)) \leq V_2(t, x(t))$ for all $t \in I$.

First we shall show that if $\ell_1 V_1(t, x(t)) < V_2(t, x(t))$ for some $t = \tau \in J$, then there exists $\mu \in (\tau, \omega_+)$ such that

$$\ell_1 V_1(t, x(t)) < V_2(t, x(t)) \qquad (t \in [\tau, \mu]). \tag{27}$$

In fact, if $\tau = \tau_k$, then (27) follows by continuity. If $\tau = \tau_k$, then from $\ell_1 V_1(\tau_k, x(\tau_k)) < V_2(\tau_k, x(\tau_k))$ by (14) and (15) it follows that

$$\ell_1 V_1(\tau_k^+, x(\tau_k^+)) \leq \ell_1 V_1(\tau_k, x(\tau_k)) < V_2(\tau_k, x(\tau_k)) \leq V_2(\tau_k^+, x(\tau_k^+)),$$

which, also by continuity, implies (27) for some $\mu > \tau$.

Now we claim that if $\ell_1 V_1(\tau, x(\tau)) < V_2(\tau, x(\tau))$ for $\tau \in J$, then $\ell_1 V_1(t, x(t)) < V_2(t, x(t))$ for $t \in [\tau, \omega_+)$. Suppose that this is not true, i.e. that there exists $s > \mu$ such that $\ell_1 V_1(s, x(s)) \geq V_2(s, x(s))$. Let s_0 be the infimum of the numbers s enjoying this property. Then $s_0 \geq \mu > \tau$ and

$$\ell_1 V_1(s_0^+, x(s_0^+)) \geq V_2(s_0^+, x(s_0^+)), \tag{28}$$

$$\ell_1 V_1(t, x(t)) < V_2(t, x(t)) \qquad (t \in [\tau, s_0)), \tag{29}$$

whence by continuity from the left

$$\ell_1 V_1(s_0, x(s_0)) \leq V_2(s_0, x(s_0)). \tag{30}$$

We have that

$$V_1(s_0, x(s_0)) < \ell_2 V_2(s_0, x(s_0)). \tag{31}$$

Otherwise, $V_1(s_0, x(s_0)) \geq \ell_2 V_2(s_0, x(s_0))$ and by (30)

$$V_2(s_0, x(s_0)) \geq \ell_1 V_1(s_0, x(s_0)) \geq \ell_1 \ell_2 V_2(s_0, x(s_0)),$$

whence it follows that $V_2(s_0, x(s_0)) = 0$ and $x(s_0) = 0$ (by (30) and (6)) which is impossible.

From (31) and the continuity from the left of $x(t)$ it follows that there exists $\eta < s_0$ such that

$$V_1(t, x(t)) > \ell_2 V_2(t, x(t)) \qquad (t \in [\eta, s_0]).$$

Then in the interval $J_1 = [\eta, s_0] \cap [\tau, s_0]$ the function $V_1(t, x(t))$ is nonincreasing and the function $V_2(t, x(t))$ is nondecreasing and for $t \in J_1$ by (14), (28) and (15) we have

$$\begin{aligned} \ell_1 V_1(t, x(t)) \geq & \ell_1 V_1(s_0, x(s_0)) \geq \ell_1 V_1(s_0^+, x(s_0^+)) \geq V_2(s_0^+, x(s_0^+)) \\ \geq & V_2(s_0, x(s_0)) \geq V_2(t, x(t)), \end{aligned}$$

which contradicts (29). Thus the assertion is proved. It implies that if

$$\ell_1 V_1(t, x(t)) \geq V_2(t, x(t)) \tag{32}$$

is valid for $t = \tau$, then it is also valid for $t \in (\omega_-, \tau]$.

If the assumption $\rho = 0$ is not valid, then the assertion in relation to (32) can be proved in the same way if in the proof we replace

$V_i(t, x)$ by $\exp(\int_{t_0}^t r\rho(u)du)V_i(t, x)$, $i = 1, 2$.

As in the proof of Theorem 1, considering a sequence $\tau_m \to \omega_+$ we prove that there exists a k_1-dimensional subspace X_1 of solutions of (1) such that (32) is valid for all $t \in J$ and $x \in X_1$. From (6) and (32) we conclude that (23) is valid for each $x \in X_1$ and from (6), (10)-(15), (32) - that (24) is valid for each $x \in X_1$ with $L_1 = b_1^{1/r}[1 + (\ell_1 b_1)^{1/r}]N_1$. Analogous arguments show the existence of a k_2-dimensional subspace X_2 of solutions of (1) satisfying (25) and (26), which completes the proof.

Proof of Theorem 3. Suppose that (1) has a (μ_1, μ_2)-dichotomy and let

$$V_1(t,x) = \sup_{\tau \geq t} |X(\tau)P_1X^{-1}(t)| \exp(-\int_t^\tau \mu_1(u)du),$$

$$V_2(t,x) = \sup_{\tau < t} |X(\tau)P_2X^{-1}(t)| \exp(-\int_t^\tau \mu_2(u)du),$$

for each $(t,x) \in J \times S^n$, where $X(t)$ and P_i are as in (3) and (4).

First we shall show that the relations (5), (6) hold with $r = 1$ and $Q_i(t) = X(t)P_iX^{-1}(t)$, $i = 1,2$. In fact, (4) implies immediately that $|Q_i(t)| \leq M_i$, $t \in J$. From the definitions of $V_i(t,x)$, $i = 1,2$ and the continuity from the left of $X(\tau)$ it follows that

$$|Q_i(t)x| = |X(t)P_iX^{-1}(t)x| \leq V_i(t,x), \quad i = 1,2$$

and from (4) with $\xi = X^{-1}(t)x$ we have

$$\begin{aligned}|X(\tau)P_iX^{-1}(t)x| \leq& L_i \exp(\int_t^\tau \mu_i(u)du|X(t)P_iX^{-1}(t)x| \\ =& L_i \exp(\int_t^\tau \mu_i(u)du)|Q_i(t)x| \quad ((-1)^i(t-\tau) \geq 0).\end{aligned}$$

That is why

$$V_i(t,x) \leq L_i|Q_i(t)x|, \qquad i = 1,2$$

with which (5), (6) are proved.

For $t \in J$ and $x, y \in S^n$ we have

$$\begin{aligned}&|V_1(t,x) - V_1(t,y)| \\ =& |\sup_{\tau \geq t} |X(\tau)P_1X^{-1}(t)x|e^{-\int_t^\tau \mu_1} - \sup_{\tau \geq t} |X(\tau)P_1X^{-1}(t)y|e^{-\int_t^\tau \mu_1}| \\ \leq& \sup_{\tau \geq t} |X(\tau)P_1X^{-1}(t)(x-y)|e^{-\int_t^\tau \mu_1} \\ =& V_1(t, x-y) \leq L_1|Q_1(t)(x-y)| \leq L_1M_1|x-y|,\end{aligned}$$

i.e. $V_1(t,x)$ is Lipschitz continuous in x. Analogously it is proved that $V_2(t,x)$ is also Lipschitz continuous in x.

Let $t \in (\tau_k \tau_{k+1})$, $x \in S^n$ and $0 < \delta < \min(\tau_{k+1} - t, t - \tau_k)$. Then

$$\begin{aligned}|V_1(t+\delta,y) - V_1(t,x)| \leq& |V_1(t+\delta,y) - V_1(t+\delta,x)| \\ &+ |V_1(t+\delta,x) - V_1(t+\delta, X(t+\delta, X(t+\delta)X^{-1}(t)x)| \\ &|V_1(t+\delta, X(t+\delta)X^{-1}(t)x) - V_1(t,x)|.\end{aligned} \tag{33}$$

The first two addends in (33) are small when δ and $|x-y|$ are small since $V_1(t,x)$ is Lipschitz continuous in x. If for $\delta \geq 0$ we set

$$a(\delta) = \sup_{\tau \geq t+\delta} |X(\tau)P_1X^{-1}(t)x|e^{-\int_t^\tau \mu_1}$$

then a straightforward verification shows that

$$|V_1(t+\delta, X(t+\delta)X^{-1}(t)x) - V_1(t,x)| = |a(\delta)e^{\int_t^{t+\delta}\mu_1} - a(0)|. \tag{34}$$

Since the function $a(\delta)$ is nonincreasing for $\delta \geq 0$ and $a(\delta) \to a(0)$ as $\delta \to 0+$, then (33) and (34) imply the continuity of $V_1(t,x)$ in the set G_k, $k \in \mathbb{Z}$. Analogously the continuity of $V_2(t,x)$ in G_k, $k \in \mathbb{Z}$ is proved.

Let $x(t)$ be a solution of (1) and $h > 0$. Then for $t \neq \tau_k$

$$\begin{aligned} V_1(t+h, x(t+h)) &= \sup_{\tau \geq t+h} |X(\tau)P_1X^{-1}(t+h)x(t+h)|e^{-\int_{t+h}^{\tau}\mu_1} \\ &= \sup_{\tau \geq t+h} |X(\tau)P_1X^{-1}(t)x(t)|e^{-\int_{t+h}^{\tau}\mu_1} \\ &\leq \sup_{\tau \geq t} |X(\tau)P_1X^{-1}(t)x(t)|e^{-\int_t^{\tau}\mu_1} \cdot e^{\int_t^{t+h}\mu_1} \\ &= V_1(t,x(t))e^{\int_t^{t+h}\mu_1}. \end{aligned}$$

therefore,

$$\frac{1}{h}[V_1(t+h), x(t+h)) - V_1(t,x(t))] \leq \frac{1}{h}[e^{\int_t^{t+h}\mu_1} - 1]V_1(t,x(t)),$$

i.e. $D^+V_1(t,x(t)) \leq \mu_1(t)V_1(t,x(t))$ which implies $\dot{V}_1(t,x) \leq \mu_1(t)V_1(t,x)$ since $V_1(t,x)$ is Lipschitz continuous in x. Analogously we find

$$D_-V_2(t,x(t)) \geq \mu_2(t)V_2(t,x(t)),$$

which implies $D^+V_2(t,x(t)) \geq \mu_2(t)V_2(t,x(t))$ since $V_2(t,x(t))$ and $\mu_2(t)$ are continuous for $t \neq \tau_k$. Thus

$$\dot{V}_2(t,x) \geq \mu_2(t)V_2(t,x)$$

with which (16) and (17) are proved.

Now we shall prove the existence of the limits $V_i(\tau_k^+, x)$ and $V_i(\tau_k^- x)$, $i = 1,2$. Let $t_i \in (\tau_k, \tau_{k+1})$, $x_i \in S^n$, $u_i = X(t_i)X^{-1}(\tau_k^+)x$, $i = 1,2$. Then

$$\begin{aligned} |V_1(t_1,x_1) - V_2(t_2,x_2)| \leq & |V_1(t_1,x_1) - V_1(t_1,u_1)| + |V_1(t_2,x_2) - V_1(t_2,u_2)| \\ & + |V_1(t_1,u_1) - V_1(t_2,u_2)|. \end{aligned} \tag{35}$$

By the Lipschitz continuity of $V_1(t,x)$ in x

$$|V_1(t_i,x_i) - V_1(t_i,u_i)| \leq L_1|x_i - u_i| \leq L_1(|x_i - x| + u_i - x|).$$

But $|u_i - x| = |X(t_i)X^{-1}(\tau_k^+)x - x| \to 0$ as $t_i \to \tau_k^+$. Therefore, the first two addends in (35) tend to zero as $(t_i, x_i) \to (\tau_k^+, x)$, $i = 1, 2$. Moreover, if for $\delta > 0$ we define

$$a(\delta) = \sup_{\tau \geq \tau_k + \delta} |X(\tau)P_1X^{-1}(\tau_k)x|e^{-\int_{\tau_k}^{\tau} \mu_1}$$

then

$$\begin{aligned} &|V_1(t_1, u_1) - V_1(t_2, u_2)| \\ =&|\sup_{\tau \geq t_1} |X(\tau)P_1X^{-1}(t_1)X(t_1)X^{-1}(\tau_k^+)x|e^{-\int_{t_1}^{\tau} \mu_1} \\ &- \sup_{\tau \geq t_2} |X(\tau)P_1X^{-1}(t_2)X(t_2)X^{-1}(\tau_k^+)x|e^{-\int_{t_2}^{\tau} \mu_2}| \\ =&|a(t_1 - \tau_k)e^{\int_{\tau_k}^{t_1} \mu_1} - a(t_2 - \tau_k)e^{\int_{\tau_k}^{t_2} \mu_1}|, \end{aligned}$$

i.e. the third addended in (35) tends to zero as $t_i \to \tau_k^+$, $i = 1, 2$. All this shows that the limit $V_1(\tau_k^+, x)$ exists. The existence of the other limits is proved analogously.

Now we can calculate

$$\begin{aligned} V_1(\tau_k^+, A_kx) &= \lim_{\nu \to \tau_k^+} V_1(\nu, X(\nu)X^{-1}(\tau_k^+)A_kx) \\ &= \lim_{\nu \to \tau_k^+} \sup_{\tau \geq \nu} |X(\tau)P_1X^{-1}(\nu)X(\nu)X^{-1}(\tau_k^+)A_kx|e^{-\int_{\nu}^{\tau} \mu_1} \\ &= \lim_{\nu \to \tau_k^+} \sup_{\tau \geq \nu} |X(\tau)P_1X^{-1}(\tau_k)x|e^{-\int_{\nu}^{\tau} \mu_1} \\ &= \sup_{\tau > \tau_k} |X(\tau)P_1X^{-1}(\tau_k)x|e^{-\int_{\tau_k}^{\tau} \mu_1} \leq V_1(\tau_k, x), \\ V_1(\tau_k^-, x) &= \lim_{\lambda \to \tau_k^-} V_1(\lambda, X(\lambda)X^{-1}(\tau_k)x) \\ &= \sup_{\tau \geq \tau_k} |X(\tau)P_1X^{-1}(\tau_k)x|e^{-\int_{\tau_k}^{\tau} \mu_1} = V_1(\tau_k, x), \\ V_2(\tau_k^+, x) &= \lim_{\nu \to \tau_k^+} V_2(\nu, X(\nu)X^{-1}(\tau_k^+)A_kx) \\ &= \sup_{\tau \leq \tau_k} |X(\tau)P_2X^{-1}(\tau_k)x|e^{-\int_{\tau_k}^{\tau} \mu_2} \geq V_2(\tau_k, x), \\ V_2(\tau_k^-, x) &= \lim_{\lambda \to \tau_k^-} V_2(\lambda, X(\lambda)X^{-1}(\tau_k)x) \\ &= \sup_{\tau < \tau_k} |X(\tau)P_2X^{-1}(\tau_k)x|e^{-\int_{\tau_k}^{\tau} \mu_2} = V_2(\tau_k, x). \end{aligned}$$

Hence $V_i(t, x) \in V_0$, $i = 1, 2$ and (18), (19) are valid. Thus we completed the proof of Theorem 3.

Theorem 4. *Let the matrix-valued functions $H_i(t) \in PC(J, S^n)$, $i = 1, 2$ be Hermitian for each $t \in J$ and have derivatives $H_i'(t) \in PC(J, S^n)$, $i = 1, 2$. Let there exist constants $\ell_i \geq 0$, $b_i \geq 0$, $i = 1, 2$ such that $0 \leq \ell_i b_i < 1$ and for any $t \in J$:*

(i) $H_1(t)H_2(t) = 0$,

(ii) $H_1(t) + H_2(t) \geq I$,

(iii) $H_i(t) \leq b_i I$, $i = 1, 2$,

(iv) $H(\lambda; t) = \lambda_1 H_1(t) - \lambda_2 H_2(t)$ *satisfies*

$$H' + A^*H + HA \leq 2\mu_1 H \quad \text{if } \lambda = (1, \ell_2),\ H_1 - \ell_2 H_2 \geq 0,\ t \neq \tau_k,$$

$$H' + A^*H + HA \leq 2\mu_2 H \quad \text{if } \lambda = (\ell_1, 1),\ \ell_1 H_1 - H_2 \leq 0,\ t \neq \tau_k,$$

(v) $A_k^* H_i(\tau_k^+) A_k = H_i(\tau_k)$, $\quad i = 1, 2,\ k \in \mathbb{Z}$.

Then equation (1) *has a* (μ_1, μ_2)*-dichotomy.*

Proof. This theorem follows from Theorem 1. If $\text{rank} H_i(t) = k_i(t)$ then (i) implies nullity $H_1(t) \geq k_2(t)$ so that $k_1(t) + k_2(t) \leq n$ and (ii) imply $k_1(t) + k_2(t) \geq n$. Hence, $k_1(t) + k_2(t) = n$, which implies that k_1, k_2 are constants on each interval $(\tau_k, \tau_{k+1}]$ since these functions are lower semicontinuous on $(\tau_k, \tau_{k+1}]$, $k \in \mathbb{Z}$. But from (v) we conclude that $\text{rank} H_i(\tau_k^+) = \text{rank} H_i(\tau_k)$ and therefore k_1, k_2 are constants in J. By (i) the matrix $H_i(t)$ commutes with $H_1(t) + H_2(t)$ thus $Q_i(t) = H_i(t)[H_1(t) + H_2(t)]^{-1}$, $i = 1, 2$ are supplementary Hermitian projectors of rank k_i, $i = 1, 2$ for each $t \in J$. The functions $V_i(t, x) = x^* H_i(t) x$, $i = 1, 2$ satisfy conditions (5), (6) and the conditions of Theorem 1. We omit the proof of this assertion since it is carried out as in [4]. Proposition 2.6. We shall only note that from (v) immediately follows that $V_i(t, x)$, $i = 1, 2$ satisfy condition (g) of Theorem 1.

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