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COEFFICIENT ESTIMATES FOR BOUNDED STARLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract. Let $F(b,M,n)$ ($b \neq 0$, complex, $M > \frac{1}{2}$, and n is a positive integer) denote the class of functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ analytic in $U = \{z : |z| < 1\}$ which satisfy for fixed M, $f(z)/z \neq 0$ in U and

$$\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M, \qquad z \in U.$$

Also let $F^*(b,M,n)$ ($b \neq 0$, complex, $M \geq 1$, and n is a positive integer) denote the class of functions $f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_k z^k$ analytic in the annulus $U^* = \{z : 0 < |z| < 1\}$ which satisfy

$$\left| \frac{b - 1 - \frac{zf'(z)}{f(z)}}{b} - M \right| < M, \qquad z \in U^*.$$

In this paper we obtain bounds for the coefficients of functions of the above classes.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k; \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Let Ω denote the class of bounded analytic functions $w(z)$ in U, of the form

$$w(z) = \sum_{k=n}^{\infty} b_k z^k; \tag{1.2}$$

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satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

Let $P_M(n)$, where n is a positive integer, denote the class of functions of the form

$$P(z) = 1 + \sum_{k=n}^{\infty} c_k z^k; \tag{1.3}$$

which are analytic in U and satisfying

$$|P(z) - M| < M \tag{1.4}$$

for a fixed real M,$M > \frac{1}{2}$.

It is easy to show that

$$P(z) = \frac{1 + w(z)}{1 - Qw(z)}, \quad Q = 1 - \frac{1}{M}, M > \frac{1}{2}, w \in \Omega; \tag{1.5}$$

is a function in $P_M(n)$.

For $f(z) \in A$, we say that $f(z)$ belongs to the class $F(b, M, n)$ ($b \neq 0$, complex, $M > \frac{1}{2}$, and n is a positive integer), of bounded starlike functions of complex order, if and only if $f(z)/z \neq 0$ in U and fixed M,

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) = P(z), \qquad z \in U \tag{1.6}$$

for some $P(z) \in P_M(n)$.

Or, equivalently

$$\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M, z \in U. \tag{1.7}$$

From (1.5) and (1.6) it follows that $f(z) \in F(b, M, n)$ if and only if for $z \in U$

$$\frac{zf'(z)}{f(z)} = \frac{1 + [b(1+Q) - Q]w(z)}{1 - Qw(z)}, \quad w \in \Omega, \quad Q = 1 - \frac{1}{M}. \tag{1.8}$$

We note that by giving specific values to b, M and n, we obtain the following important subclasses studied by various authors in earlier papers:

(i) For $b = (1-a)e^{-i\beta}\cos\beta$, $0 \leq a < 1, |\beta| < \frac{\pi}{2}$ and $M = \frac{\sigma}{1-a} > \frac{1}{2}$, $F((1-a)e^{-i\beta}\cos\beta, \frac{\sigma}{1-a}, n) = S_\beta(a, \sigma, n)$ (Goplakrishna and Shetiya [8]);

(ii) For $n = 1$, $b = 1 - \alpha$, $0 \leq \alpha < 1$, and M tending to ∞, $F(1-\alpha, \infty, 1) = S^*(\alpha)$ (Robertson [19]);

(iii) For $n = 1, b = (1-\alpha)e^{-i\lambda}\cos\lambda, 0 \leq \alpha < 1, |\lambda| < \frac{\pi}{2}$, and M tending to ∞, $F((1-\alpha)e^{-i\lambda}\cos\lambda, \infty, 1) = S^\lambda(\alpha)$ (Libera [12]);

(iv) For $n = 1$ and $b = e^{-i\lambda}\cos\lambda$, $|\lambda| < \frac{\pi}{2}$, $F(e^{-i\lambda}\cos\lambda, M, 1) = F_{\lambda,M}$(Kulshretha [11]);

(v) For $n = 1$ and M tending to ∞, $F(b, \infty, 1) = S(1-b)$ (Nasr and Aouf [15]);

(vi) For $n=1, F(b,M,1)=F(b,M)$ (Nasr and Aouf [16]);

(vii) For $n=1$ and $b=(1-\alpha)e^{-i\lambda}\cos\lambda, 0\le\alpha<1, |\lambda|<\frac{\pi}{2}, F((1-\alpha)e^{-i\lambda}\cos\lambda, M, 1)=F_M(\lambda,\alpha)$ (Aouf [1,3]).

Let $F^*(b,M,n)$ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=n}^{\infty}a_k z^k; \tag{1.9}$$

which are analytic in the punctured disc $U^*=\{z: 0<|z|<1\}$, and satisfying

$$1-\frac{1}{b}(\frac{zf'(z)}{f(z)}+1)=P(z), \quad z\in U^* \tag{1.10}$$

for some $P(z)\in P_M(n+1)$.

Or, equivalently

$$\left|\frac{b-1-\frac{zf'(z)}{f(z)}}{b}-M\right|<M, \quad M\ge 1, \quad z\in U^*. \tag{1.11}$$

Thus from (1.7), (1.11) and (1.8) it follows that $f(z)\in F^*(b,M,n)$ if and only if

$$\text{(i)} \qquad \frac{1}{f(z)}\in F(b,M,n) \tag{1.12}$$

$$\text{(ii)} \qquad -\frac{zf'(z)}{f(z)}=\frac{1+[b(1+Q)-Q]w(z)}{1-Qw(z)}, w\in\Omega, \quad Q=1-\frac{1}{M}. \tag{1.13}$$

Also we note that by giving specific values to b, M and n, we obtain the following important subclasses studied by various authors in earlier papers:

(i) For $b=(1-a)e^{-i\beta}\cos\beta, 0\le a<1, |\beta|<\frac{\pi}{2}$ and $M=\frac{\sigma}{1-a}\ge 1$, $F^*((1-a)^{-i\beta}\cos\beta, \frac{\sigma}{1-a}, n)\doteq U_\beta(a,\sigma,n)$ (Goplakrishna and Shetiya [8]);

(ii) For $n=b=1$ and M tending to $\infty, F^*(1,\infty,1)=F^*(1)$ (Clunie [5]);

(iii) For $n=1, b=(1-a), 0\le a<1$ and M tending to $\infty, F^*(1-a,\infty,1)=F^*(1-a)$ (Pommerenke [18] and Kaczmarski [10]);

(iv) For $n=1, b=(1-a)e^{-i\beta}\cos\beta, 0\le a<1, |\beta|<\frac{\pi}{2}$ and M tending to $\infty, F^*((1-\alpha)\cdot e^{-i\beta}\cos\beta,\infty,1)=F^*(a,\beta)$ (Kaczmarski [10]);

(v) For $n=1$ and $b=1-a, 0\le a<1; F^*(1-a,M,1)=F^*_M(a)$ (Kaczmarski [10]);

(vi) For $n=1$ and $b=(1-a)e^{-i\beta}\cos\beta, 0\le a<1, |\beta|<\frac{\pi}{2}, F((1-a)e^{-i\beta}\cos\beta, M, 1)=F^*_M(a,\beta)$ (Kaczmarski [10]);

(vii) For $n=1, F^*(b,M,1)=F^*(b,M)$ (Aouf [2]);

(viii) For $n=1$ and M tending to $\infty, F^*(b,\infty,1)=F^*(b)$ (Aouf [2]).

In this paper we obtain bounds for the coefficients of functions of the classes $F(b,M,n)$, $M>\frac{1}{2}$ and $F^*(b,M,n)$, $M\ge 1$.

2. Coefficient estimates for the classes $F(b, M, n)$ and $F^*(b, M, n)$.

We shall use the following lemma in our investigation:

Lemma 1. *If $Q \neq 1$ and n and q are positive integers, then*

$$(1-Q^2)\left\{(\frac{1+Q}{1-Q})|b|^2 + \sum_{m=1}^{q}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}mn\,Re\{b\} - m^2n^2\right]\right.$$
$$\left.\cdot\left[\frac{1}{m!}\prod_{j=0}^{m-1}u_j\right]^2\right\} = \left\{\frac{n}{q!}\prod_{j=0}^{q}u_j\right\}^2, \tag{2.1}$$

where

$$u_j = \left|(\frac{1+Q}{n})b + jQ\right|, \quad j = 0, 1, 2, \cdots. \tag{2.2}$$

The lemma can be proved by induction on q in the same way as the lemma in [7].

Theorem 1. *If the function $f(z)$ defined by (1.1) is in the class $F(b, M, n)$, $M > \frac{1}{2}$ and $Q \neq 1$, and if*

$$n^2 - \frac{2Q}{1-Q}nRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 \geq 0, \tag{2.3}$$

then

$$\sum_{k=mn+1}^{(m+1)n}(k-1)^2|a_k|^2 \leq (1+Q)^2|b|^2, \quad m = 1, 2, \cdots. \tag{2.4}$$

If, on the other hand,

$$n^2 - \frac{2Q}{1-Q}nRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 < 0, \tag{2.5}$$

then

$$\sum_{k=mn+1}^{(m+1)n}(k-1)^2|a_k|^2 \leq \left\{\frac{n}{(m-1)!}\prod_{j=0}^{m-1}u_j\right\}^2 \tag{2.6}$$

for $m = 1, 2, \cdots, q_0 + 1$, and

$$\sum_{k=mn+1}^{(m+1)n}(k-1)^2|a_k|^2 \leq \left\{\frac{n}{q_0!}\prod_{j=0}^{q_0}u_j\right\}^2 \tag{2.7}$$

for $m = q_0 + 2, q_0 + 3, \cdots$, where u_j is given by (2.2) and q_0 is the natural number determined by $q_0 \in [c-1, c)$, where

$$c = \frac{(\frac{Q}{1-Q})Re\{b\} + \sqrt{(\frac{Q}{1-Q}Re\{b\})^2 + (\frac{1+Q}{1-Q})|b|^2}}{n}. \tag{2.8}$$

Also

$$\sum_{k=n+1}^{\infty}\left[(k-1)^2-(\frac{1+Q}{1-Q})|b|^2-\frac{2Q}{1-Q}(k-1)Re\{b\}\right]|a_k|^2\le(\frac{1+Q}{1-Q})|b|^2. \quad (2.9)$$

The estimates in (2.4) *and* (2.6) *are sharp.*

Proof. Since $f(z)\in F(b,M,n)$, (1.8) gives

$$[zf'(z)-f(z)]=\{[(1+Q)b-Q]f(z)+Qzf'(z)\}\,w(z).$$

Substituting the series expansions of $f(z)$ and $w(z)$, we obtain

$$\sum_{k=n+1}^{\infty}(k-1)a_kz^k = \left\{(1+Q)bz+\sum_{k=n+1}^{\infty}[(1+Q)b+Q(k-1)]\,a_kz^k\right\}\sum_{k=n}^{\infty}b_kz^k. \quad (2.10)$$

We now proceed by a method introduced by Clunie [5].

Equating the coefficients of z^k on the two sides of (2.10) for $k=n+1,\cdots,2n$, we obtain

$$(k-1)a_k=(1+Q)bb_{k-1} \quad \text{for} \quad k=n+1,\cdots,2n.$$

Therefore,

$$\sum_{k=n+1}^{2n}(k-1)^2|a_k|^2\le(1+Q)^2|b|^2\sum_{k=n+1}^{2n}|b_{k-1}|^2 \le(1+Q)^2|b|^2, \quad (2.11)$$

since, we have, for $0<r<1$,

$$\sum_{k=n}^{\infty}|b_k|^2r^{2k}=\frac{1}{2\pi}\int_0^{2\pi}|w(re^{i\varphi})|^2d\varphi\le 1$$

and letting r tends to 1 we obtain $\sum_{k=n}^{\infty}|b_k|^2\le 1$.

Again, for $p\ge n+1$, (2.10) can be put in the form

$$G(z)=H(z)w(z), \quad z\in U$$

with

$$G(z)=\sum_{k=n+1}^{n+p}(k-1)a_kz^k+\sum_{k=n+p+1}^{\infty}d_kz^k$$

and

$$H(z)=(1+Q)bz+\sum_{k=n+1}^{p}\left[(1+Q)b+Q(k-1)\right]a_kz^k,$$

where $\sum_{k=n+p+1}^{\infty}d_kz^k$ converges in U.

Since $|w(z)|<1$ for $z\in U$, we obtain, for $0<r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}|G(re^{i\varphi})|^2d\varphi \leq \frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\varphi})|^2d\varphi,$$

so that

$$\sum_{k=n+1}^{n+p}(k-1)^2|a_k|^2r^{2k}+\sum_{k=n+p+1}^{\infty}|d_k|^2r^{2k}$$
$$\leq (1+Q)^2|b|^2r^2+\sum_{k=n+1}^{p}\left|(1+Q)b+Q(k-1)|^2|a_k\right|^2r^{2k}.$$

Letting r tends to 1 and rearranging, we obtain for $p\geq n+1$,

$$\sum_{k=p+1}^{p+n}(k-1)^2|a_k|^2\leq(1-Q^2)\left\{(\frac{1+Q}{1-Q})|b|^2+\sum_{k=n+1}^{p}\left[(\frac{1+Q}{1-Q})|b|^2+\right.\right.$$
$$\left.\left.+\frac{2Q}{1-Q}(k-1)Re\{b\}-(k-1)^2\right]|a_k|^2\right\}. \tag{2.12}$$

Let the inequality (2.3) hold true. Then, for $k\geq n+1$, if $Q\cdot Re\{b\}>0$, we have

$$(\frac{1+Q}{1-Q})|b|^2+\frac{2Q}{1-Q}(k-1)Re\{b\}-(k-1)^2$$
$$=(k-1)^2\left[(\frac{1+Q}{1-Q})\frac{|b|^2}{(k-1)^2}+\frac{2Q}{1-Q}\frac{Re\{b\}}{(k-1)}-1\right]$$
$$\leq(k-1)^2\left[(\frac{1+Q}{1-Q})\frac{|b|^2}{n^2}+\frac{2Q}{1-Q}\frac{Re\{b\}}{n}-1\right]$$
$$=\frac{(k-1)^2}{n^2}\left[(\frac{1+Q}{1-Q})|b|^2+\frac{2Q}{1-Q}nRe\{b\}-n^2\right]$$
$$\leq 0.$$

If $Q\cdot Re\{b\}<0$, we have

$$(\frac{1+Q}{1-Q})|b|^2+\frac{2Q}{1-Q}(k-1)Re\{b\}-(k-1)^2$$
$$\leq(\frac{1+Q}{1-Q})|b|^2+\frac{2Q}{1-Q}\cdot n\cdot Re\{b\}-n^2$$
$$\leq 0.$$

Hence (2.12) yields

$$\sum_{k=p+1}^{p+n}(k-1)^2|a_k|^2 \le (1+Q)^2|b|^2 \quad \text{for} \quad p \ge n+1.$$

Putting $p = mn, m \ge 2$ and combining with (2.11) we obtain (2.4).

Suppose now the inequality (2.5) is true. Let q_0 be as defined in the statement of the theorem. Then q_0 is the largest of the natural numbers k for which

$$k^2n^2 - \frac{2Q}{1-Q}knRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 < 0.$$

We now establish by an inductive argument inequalities (2.6) for $m = 1, 2, \cdots, q_0+1$ and the inequalities

$$\sum_{k=mn+1}^{(m+1)n}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k-1)Re\{b\} - (k-1)^2\right]|a_k|^2$$
$$\le \left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}mnRe\{b\} - m^2n^2\right]\left[\frac{1}{m!}\prod_{j=0}^{m-1}u_j\right]^2 \tag{2.13}$$

for $m = 1, \cdots, q_0$.

For $m = 1$, (2.6) reduces to (2.11) whereas the left member of (2.13)

$$\le \left[\frac{(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}nRe\{b\} - n^2}{n^2}\right]\sum_{k=n+1}^{2n}(k-1)^2|a_k|^2$$
$$\le \left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}nRe\{b\} - n^2\right](\frac{1+Q}{n})^2|b|^2$$

by (2.11), so that (2.13) holds for $m = 1$.

Suppose that (2.6) and (2.13) hold for $m = 1, \cdots, q-1$, where $2 \le q \le q_0$. For $p = qn$, (2.12) yields

$$\sum_{k=qn+1}^{(q+1)n}(k-1)^2|a_k|^2 \le (1-Q)^2\left\{(\frac{1+Q}{1-Q})|b|^2\right.$$
$$\left.+\sum_{m=1}^{q-1}\sum_{k=mn+1}^{(m+1)n}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k-1)Re\{b\} - (k-1)^2\right]|a_k|^2\right\}$$
$$\le (1-Q^2)\left\{(\frac{1+Q}{1-Q})|b|^2 + \sum_{m=1}^{q-1}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}mnRe\{b\} - m^2n^2\right]\cdot\right.$$
$$\left.\cdot\left[\frac{1}{m!}\prod_{j=0}^{m-1}u_j\right]^2\right\},$$

by (2.13) for $m = 1, \cdots, q-1$.

Hence by Lemma 1,

$$\sum_{k=qn+1}^{(q+1)n} (k-1)^2|a_k|^2 \leq \left[\frac{n}{(q-1)!}\prod_{j=0}^{q-1} u_j\right]^2$$

so that (2.6) holds for $m = q$. Now,

$$\begin{aligned}
&\sum_{k=qn+1}^{(q+1)n} \left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k-1)Re\{b\} - (k-1)^2\right]|a_k|^2 \\
&\leq \left[\frac{(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}nRe\{b\} - q^2n^2}{q^2n^2}\right]\sum_{k=qn+1}^{(q+1)n}(k-1)^2|a_k|^2 \\
&\leq \left[\frac{(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}qnRe\{b\} - q^2n^2}{q^2n^2}\right]\left[\frac{n}{(q-1)!}\prod_{j=0}^{q-1} u_j\right]^2,
\end{aligned}$$

using (2.6) with $m = q$ (since $(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}qnRe\{b\} - q^2n^2 > 0$ because $q \leq q_0$). Thus (2.13) holds for $m = q$.

Hence (2.6) and (2.13) hold for $m = 1, \cdots, q_0$. It follows now, by the argument used above to show that (2.6) holds for $m = q$, that (2.6) holds for $m = q_0 + 1$.

By the definition of q_0, we have

$$(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}n(q_0+1)Re\{b\} - (q_0+1)^2n^2 \leq 0.$$

Hence $(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k-1)Re\{b\} - (k-1)^2 \leq 0$ for $k > (q_0+1)n$. Thus, for $p \geq (q_0+1)n$, (2.12) yields

$$\begin{aligned}
&\sum_{k=p+1}^{p+n}(k-1)^2|a_k|^2 \\
&\leq (1-Q^2)\left\{(\frac{1+Q}{1-Q})|b|^2 + \sum_{m=1}^{q_0}\sum_{k=mn+1}^{(m+1)n}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k-1)Re\{b\} - (k-1)^2\right]|a_k|^2\right\} \\
&\leq (1-Q^2)\left\{(\frac{1+Q}{1-Q})|b|^2 + \sum_{m=1}^{q_0}\left[(\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}mnRe\{b\} - m^2n^2\right]\left[\frac{1}{m!}\prod_{j=0}^{m-1}u_j\right]^2\right\},
\end{aligned}$$

by (2.13) for $m = 1, 2, \cdots, q_0$.

Hence, by Lemma 1,

$$\sum_{k=p+1}^{p+n}(k-1)^2|a_k|^2 \le \left[\frac{n}{q_0!}\prod_{j=0}^{q_0}u_j\right] \quad \text{for } p \ge (q_0+1)n.$$

Putting $p = mn, m = q_0+2, q_0+3, \cdots$, we obtain (2.7).

Finally, we obtain from (2.12),

$$\sum_{k=n+1}^{p}\left[(k-1)^2-(\frac{1+Q}{1-Q})|b|^2-\frac{2Q}{1-Q}(k-1)Re\{b\}\right]|a_k|^2 \le (\frac{1+Q}{1-Q})|b|^2$$

for $p \ge n+1$ and letting p tends to ∞, we obtain (2.9). This completes the proof of Theorem 1.

The estimates (2.4) and (2.6) are sharp with equality holding in (2.4) for a given m for the fucnction $f_\varepsilon(z)(|\varepsilon|=1)$ defined by

$$f_\varepsilon(z) = \begin{cases} z[1-\varepsilon Q z^{mn}]^{-(\frac{1+Q}{Q})\frac{b}{mn}}, & Q \ne 0,\\ z\exp[\frac{\varepsilon b z^{mn}}{mn}], & Q = 0, \end{cases}$$

if $n^2 - \frac{2Q}{1-Q}nRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 \ge 0$.

Also the equality in (2.6) holds for the function $f_\varepsilon(z)(|\varepsilon|=1)$ defined by

$$f_\varepsilon(z) = \begin{cases} [1-\varepsilon Q z^{n}]^{-(\frac{1+Q}{Q})\frac{b}{n}}, & Q \ne 0,\\ z\exp[\frac{\varepsilon b z^{n}}{n}], & Q = 0, \end{cases}$$

if $n^2 - \frac{2Q}{1-Q}nRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 < 0$.

Remark 1. Under the hypothesis of Theorem 1, since Q tends to 1 as M tends to ∞, it follows that $n^2 - \frac{2Q}{1-Q}nRe\{b\} - (\frac{1+Q}{1-Q})|b|^2 < 0$ for all sufficiently large M and hence (2.6) and (2.7) hold for all sufficiently large M. Also since q_o tends to $\infty, 1+Q$ tends to 2 as M tends to ∞, we obtain the following corollaries:

Corollary 1. *If* $f(z) = z+\sum_{k=n+1}^{\infty}a_k z^k \in F((1-\alpha)e^{-i\lambda}\cos\lambda, M, n) = F_M(\lambda,\alpha,n)$, $|\lambda| < \frac{\pi}{2}$ *and* $0 \le \alpha < 1$, *then*

$$\sum_{k=mn+1}^{(m+1)n}(k-1)^2|a_k|^2 \le \left\{\frac{n}{(m-1)!}\prod_{j=0}^{m-1}\left|(\frac{1+Q}{n})(1-\alpha)e^{-i\lambda}\cos\lambda + jQ\right|\right\}^2$$

for $m = 1, 2, \cdots$. *The estimates are sharp for the function* $f_\varepsilon(z)(|\varepsilon|=1)$ *given by*

$$f_\varepsilon(z) = \begin{cases} z[1-\varepsilon Q z^{n}]^{-(\frac{1+Q}{Q})\frac{(1-\alpha)e^{-i\lambda}\cos\lambda}{n}}, & Q \ne 0,\\ z\exp[\frac{\varepsilon(1-\alpha)e^{-i\lambda}z^n\cos\lambda}{n}], & Q = 0. \end{cases}$$

Corollary 2. *If* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in F(b, \infty, n) = S(1-b, n)$, *then*

$$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \le \left\{ \frac{n}{(m-1)!} \prod_{j=0}^{m-1} \left| \frac{2b}{n} + j \right| \right\}^2$$

for $m = 1, 2, \cdots$. *The estimates are sharp for the function* $f_\varepsilon(z)(|\varepsilon| = 1)$ *given by*

$$f_\varepsilon(z) = (1 - \varepsilon z^n)^{\frac{-2b}{n}}.$$

Corollary 3 [14]. *If* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in F(e^{-i\lambda}\cos\lambda, \frac{1}{\cos\lambda}, n) = (H^*)_n^\lambda$, *then*

$$|a_k| \le \frac{(2 - \cos\lambda)\cos\lambda}{k-1}, \qquad k \ge n+1.$$

The estimates are sharp for each $k \ge n+1$.

Corollary 4 [7,8,14]. *If* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in F((1-\alpha)e^{-i\lambda}\cos\lambda, \infty, n) = S^\lambda(\alpha, n), 0 \le \alpha < 1, |\lambda| < \frac{\pi}{2}$, *then*

$$\sum_{k=mn+1}^{(m+1)n} (k-1)^2 |a_k|^2 \le \left\{ \frac{n}{(m-1)!} \prod_{j=0}^{m-1} \left| \frac{2(1-\alpha)e^{-i\lambda}\cos\lambda}{n} + j \right| \right\}^2$$

for $m = 1, 2, \cdots$. *The estimates are sharp.*

Remark 2.

(1) Choosing (i) $\alpha = \lambda = 0$, (ii) $\lambda = 0$ in Corollary 4 we get, respectively, results of MacGregor [13] and Boyd [4].

(2) Choosing (i) $n = 1$, (ii) $n = 1, \alpha = 0$ in Corollary 1 we get, respectively, results of Aouf [1] and Kulshrestha [11].

(3) Choosing $n = 1$ in Corollary 2 we get results of Nasr and Aouf [15].

(4) Choosing $n = 1$ in Corollary 3 we get a result of Goel [6].

(5) Choosing $n = 1, b = (1-\alpha)e^{-i\lambda}\cos\lambda, 0 \le \alpha < 1, |\lambda| < \frac{\pi}{2}, M \ge 1$ in Theorem 1 we obtain a result of Plaskota [17].

Theorem 2. *If the function* $f(z)$ *defined by* (1.9) *is in the class* $F^*(b, M, n)$ *and* $M \ge 1$, *then*

$$\sum_{k=mn}^{(m+1)n-1} (k+1)^2 |a_k|^2 \le (1+Q)^2 |b|^2, \quad Q = 1 - \frac{1}{M}, \tag{2.14}$$

for $m = 1, 2, \cdots$, *and*

$$\sum_{k=n}^{\infty} \left[(k+1)^2 - (\frac{1+Q}{1-Q})|b|^2 + \frac{2Q}{1-Q}(k+1)Re\{b\} \right] |a_k|^2 \le (\frac{1+Q}{1-Q})|b|^2. \tag{2.15}$$

Estimate (2.14) *is sharp with equality holding for given* m *for the function*

$$f_\varepsilon(z) = \begin{cases} z^{-1}[1+\varepsilon Q z^{mn+1}]^{-(\frac{1+Q}{Q})\frac{b}{mn+1}}, & Q \neq 0, \\ z^{-1}\exp[\frac{-\varepsilon b z^{mn+1}}{mn+1}], & Q = 0, \end{cases}$$

where $|\varepsilon| = 1$.

The proof is analogous to that of Theorem 1 and is omitted.

Remark 3. Choosing $n = 1, b = (1-a)e^{-i\beta}\cos\beta, 0 \le a < 1, |\beta| < \frac{\pi}{2}, M = m \ge 1$, the above theorems yields results of Jakubouski [9].

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