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A SPHERICAL MAPPING AND BORSUK CONJECTURE IN RIEMANNIAN AND NON-EUCLIDEAN SPACES

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Abstract. We introduce an analog of the spherical mapping for convex bodies in a Riemannian n-manifold, and then use this construction to prove the Borsuk conjecture for some types of such bodies. The Borsuk conjecture is that each bounded set X in the Euclidean n-space can be covered by $n+1$ sets of smaller diameter. The conjecture was disproved recently by Kahn and Kalai. However Hadwiger proved the Borsuk conjecture under the additional assumption that the set X is a smooth convex body. Here we extend this result to convex bodies in Riemannian manifolds under some further restrictions.

We introduce here an analog of spherical mapping for convex bodies in a Riemannian n-manifold M^n and then use this construction to prove the Borsuk conjecture for some types of such bodies. The Borsuk conjecture stated by him in 1933 is that each bounded set X in R^n can be covered by $n+1$ sets of smaller diameter. The conjecture was disproved recently by Kahn and Kalai [9]. For more on the history of the problem and its partial solutions, see [9] and the references there. In particular, Hadwiger [6,7,8] proved the Borsuk conjecture under the additional assumption that the set X is a smooth convex body. Here, in Theorem 1, we extend this result to convex bodies in Riemannian manifolds under some further restrictions.

This paper is closely related to [4]. We remind now the basic definitions from there.

The manifold M^n is supposed to be regular but not necessarily complete. A set $C \subset M^n$ will be called *definitely convex* if

(i) C is compact and each two points of C can be connected in C by a rectifiable curve;
(ii) Any shortest path in C between two points of C is a geodesic. (The path exists due to (i).);
(iii) Any geodesic segment in C is the unique shortest path in C between its ends;
(iv) Each geodesic segment in C contains no pair of conjugate points along this segment.

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Some examples of definitely convex sets can be found in the beginning of [4]. Let C be a definitely convex set. If C is homeomorphic to a ball, it will be called a *definitely convex body*. It can be shown [4, the beginning] that if the set C has an interior point, then it is a definitely convex body.

We denote by xy both a closed geodesic segment with the ends x, y and its length. The meaning will be specified in case of possible confusion.Throughout the paper, we will deal with exterior unit normals of convex bodies. For short, we call them just normals.

Let $C \subset M^n$ be a definitely convex body, $a, b \in \partial C$ and n_a, n_b be normals at a and b respectively (defined for instance in [2,§4.5]). Suppose that the parallel translation along the chord ab turns n_a into $-n_b$. Then the normals n_a and n_b are said to be *antipodal* and the points a, b *antipodes*. (Symmetry of this relation is obvious.)

Each normal n_a has at least one antipodal normal ([4, Theorem 1] but can have more. Condition (iv) above is used in proof of this Theorem 1 in [4].

We construct now our spherical mapping for the definitely convex body $C \subset M^n$. Take a point $z \in C$ and denote by $S = S^{n-1}$ the unit sphere of directions at z. Let $N \subset TM^n$ be the set of all normals of C. For $v \in N$, denote by $A(v) \subset N$ the set of all normals antipodal to v. The notation $n_x \in N$ will imply that n_x is a normal at a point $x \in \partial C$ or $\pi(n_x) = x$ where $\pi : TM^n \to M^n$ is the natural projection. Fix some $n_x \in N$ and pick up an $n_y \in A(n_x)$. Denote by o the midpoint of the chord xy. Now let $\varphi(n_x, n_y) \in S$ be the result of parallel translation of the vector n_x to the point z along the polygonal line xoz, see Fig. 1. (The shortest geodesic segments xo and oz exist and are unique due to (ii) and (iii).) Note that

$$\varphi(n_x, n_y) = -\varphi(n_y, n_x), \tag{1}$$

i.e., $\varphi(n_x, n_y)$ and $\varphi(n_y, n_x)$ are antipodes on the sphere S.
Put

$$\Phi(n_x) = \cup_{n_y \in A(n_x)} \varphi(n_x, n_y) \tag{2}$$

thus defining a mapping

$$\Phi : N \to 2^S. \tag{3}$$

In Euclidean case, the image $\Phi(v)$, $v \in N$, is always a single point of S even when v has more than one antipodal normals. We finally define the *spherical mapping* $\Omega : \partial C \to 2^S$ *for* C *with respect to* z putting

$$\Omega(x) = \cup_{\pi(v)=x} \Phi(v), x \in \partial C. \tag{4}$$

Remark 1. Let x, $y \in \partial C$ be antipodes. Then there exists a pair n_x, n_y of antipodal normals. By (4) and (2),

$$\Omega(x) \supset \Phi(n_x) \ni \varphi(n_x, n_y); \qquad \Omega(y) \supset \Phi(n_y) \ni \varphi(n_y, n_x). \tag{5}$$

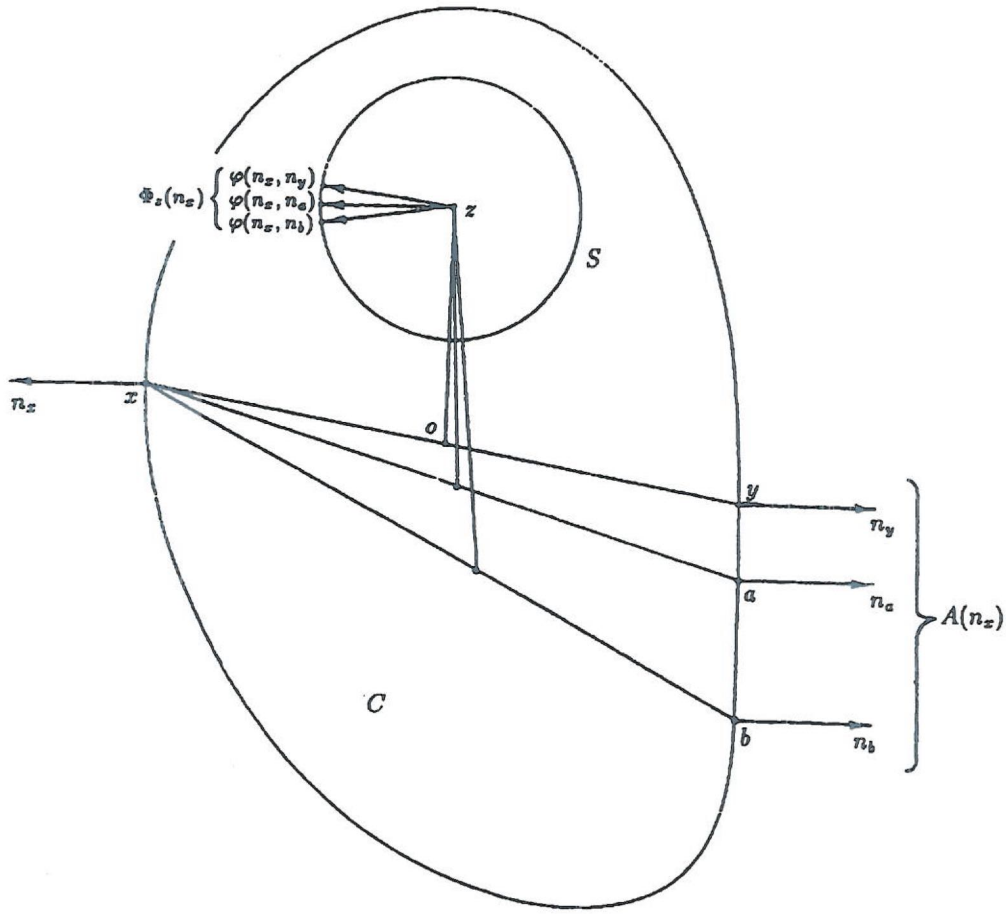

Fig. 1. The $A(n_x)$ and $\Phi_z(n_x)$. (here, each consists of three elements.)

Thus the sphere S contains a pair of antipodes, $\varphi(n_x, n_y)$ and $\varphi(n_y, n_x)$, see (1), one in $\Omega(x)$, the other in $\Omega(y)$.

Remark 2. For a set $\Delta \subset S$, put

$$\Omega^{-1}(\Delta) = \{x \in \partial C | \Omega(x) \bigcap \Delta \neq \phi\} \tag{6}$$

We would like to show that $\Omega^{-1}(\Delta)$ is closed if Δ is closed. Suppose the contrary. Then ∂C contains a sequence x_i such that

$$x_i \to_{i \to \infty} x \in \partial C, \quad \Omega(x_i) \bigcap \Delta \neq \phi, \quad \Omega(x) \bigcap \Delta = \phi. \tag{7}$$

Let $\alpha_i \in \Omega(x_i) \bigcap \Delta$. By (4), α_i lies in a set $\Phi(n_{x_i})$ and, by (2)

$$\alpha_i = \varphi(n_{x_i}, n_{y_i}) \tag{8}$$

for some normal n_{y_i} (at a point $y_i \in \partial C$) antipodal to n_{x_i}

By compactness of ∂C and N, one may assume that

$$y_i \to y \in \partial C, n_{x_i} \to n_x \in N, n_{y_i} \to n_y \in N \quad \text{as} \quad i \to \infty \tag{9}$$

where the limits n_x and n_y must be normals at x and y according to [4, Lemma 2].

Note that the shortest paths $x_i y_i$ converge to the path xy due to (ii) and (iii). Condition (iv) and regularity (of a properly selected mapping) imply now easily that n_x and n_y are antipodal. By continuity,

$$\alpha_i \to_{i\to\infty} \alpha \overset{def}{=} \varphi(n_x, n_y) \in \Phi(n_x) \subset \Omega(x). \tag{10}$$

Since Δ is closed and $\alpha_i \in \Delta$, one has $\alpha \in \Delta$. Along with (10), this means that $\alpha \in \Omega(x) \bigcap \Delta$ contrary to (7).

Remark 3. The mapping Ω appears to be only a poor surrogate of the spherical mapping in R^n. It depends on the point z and on the metric deep inside C. Moreover, even when each normal has a unique antipodal one , the normal cone at a boundary point x generally speaking is not isometric to the cone in $T_z M^n$ determined by $\Omega(x)$. However the properties of Ω established in Remarks 1 and 2 are sufficient for our objectives.

Theorem 1. *Let $C \subset M^n$ be a definitely convex body whose normal is unique at every boundary point. (I.e., each point of ∂C is regular or C is smooth.) Suppose that, for each segment $ab \subset C$ of length diam C, the direction of ab at b is a unique antipodal normal of the direction of ba at a. (As in [3, Remark 3], one can check that a, $b \in \partial C$ and both directions above are normals of C.) Then there exist compact sets $Q_i \subset C$, $i = 1, 2, \cdots, n+1$, such that $C = \cup_{i=1}^{n+1} Q_i$ and diam $Q_i <$ diam C for each i. (Thus the Borsuk conjecture holds for C.)*

Remark 4. Note that for sufficiently big sets in M^n, the Borsuk conjecture can easily fail. Consider for instance the set of 4 vertices of the standard partition of S^2 into 4 equal triangles.

Remark 5. The uniqueness of an antipodal normal assumed in Theorem 1 can easily fail even in a simple case of $M^n = H^2$ (hyperbolic plane). Consider in H^2 a triangle abc with $\angle bac = \angle acb = 40°$ and $ab = bc > ac$. Its side ab is one of its diameters. The direction n_a of the side ba at a has two antipodal normals: the direction n_b of ab at b and the vector n_c at c which forms angles 100° and 140° with the sides cb and ca respectively. The triangle in this example can be easily replaced by a smooth body.

It is not a simple task to check the uniqueness above directly. We do not know however any convenient sufficient conditons of this uniqueness which would not be too restrictive. Some rather restrictive conditions of this sort yield the following.

Theorem 2. *Let C be a smooth definitely convex body in a manifold M_k^n of constant curvature $k \in \{-1, 0, 1\}$. Suppose that*

$$\cosh(diam C) \leq 2 \quad if \quad k = -1. \tag{11}$$

then there exist compact sets $Q_i \subset C, i = 1, 2, \cdots, n+1$, such that $C = \cup_{i=1}^{n+1} Q_i$ and diam $Q_i <$ diam$C$ for each i.

Proof of Theorem 1. It follows the ideas of Hadwiger as presented in [1, §6, proof of Theorem 4]. Let $S = \cup_{i=1}^{n+1} \Delta_i$ be the standard partition of S into $n+1$ equal spherical $(n-1)$-simplexes Δ_i. Then

$$\operatorname{diam} \Delta_i < \pi, \qquad i = 1, 2, \cdots, n+1. \tag{12}$$

Put $P_i = \Omega^{-1}(\Delta_i)$. Let us show that

$$\operatorname{diam} P_i < d \overset{def}{=} \operatorname{diam} C. \tag{13}$$

(Both diameters are measured in M^n.) Suppose to the contrary that diam $P_i = d$. Since P_i is compact (see Remark 2), there exists a chord ab with a, $b \in P_i$ of the length d. The direction n_a of the chord ba at a is the unique normal of C at a and the direction n_b of the chord ab at b is the unique normal at b. Obviously n_a and n_b are antipodal. By our assumption, each of them has no other antipodal normals. Obviously $\Omega(a)$ is a point, $\Omega(b)$ is a point, and both points lie in Δ_i. By Remark 1, they are antipodes on the sphere S. This contradicts (12).

Clearly, $\partial C = \cup_{i=1}^{n+1} P_i$. Fix a point $q \in$ int C and let Q_i be the union of all the segments pq with $p \in P_i$. We leave it to the reader to show that $C = \cup_{i=1}^{n+1} Q_i$. Obviously Q_i is compact. Therefore there exists a segment xy with x, $y \in Q_i$ of length $d_i = \operatorname{diam} Q_i$. Let pq, $p \in P_i$, be a segment which contains an end of the diameter xy. By [5, Lemmas 3 and 1], the set $pq \backslash p \subset$ int C. If our end of xy differs from p then the segment xy can be extended within C still remaining a shortest path according to (iii). In this case, $d_i < d$. The only remaining possibility is that both x and $y \in P_i$. Then $d_i < d$ by (13). This completes the proof.

Proof of Theorem 2. The proof follows obviously from Theorem 1 and the following

Lemma. *Let C be a definitely convex body (not necessarily smooth) in an n-manifold M_k^n of constant curvature $k \in \{-1, 0, 1\}$. Suppose that (11) holds:* $\cosh(\mathit{diam}C) \leq 2$ *if $k = -1$. Then, for each segment $ab \subset C$ of length diam C, the direction of ab at b is a unique antipodal normal of the direction of ba at a.*

Proof. Note first that though M_k^n is not necessarily H^n, R^n, or S^n, the body C can be isometrically embedded in there by means of an "exponential type of mapping". Therefore the appropriate trigonometry can be used for the triangles within C.

Denote by n_a and n_b the two normals mentioned in the Lemma. Suppose to the contrary that a normal n_c at $c \neq b$ is antipodal to n_a. Denote by t the length of the chord ac, by α the angle $\angle bac$ and by γ the angle $\angle acb$ of the triangle abc, see Figure 2. Obviously

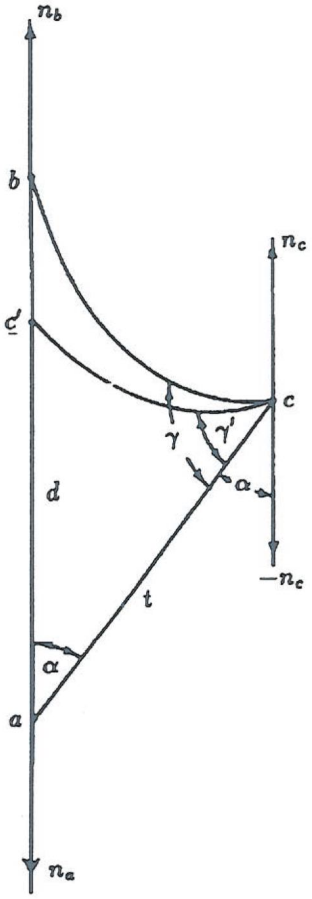

Fig. 2. The normals n_b and n_c are antipodal to n_a.

$$0 < \alpha < \pi/2; \qquad 0 < t \leq d = \operatorname{diam} C. \tag{14}$$

Denote by M_k^2 the plane of the triangle abc. Obviously the vector n_a is tangent to M_k^2. The vector $-n_c$, being the result of the parallel translation of n_a along the segment $ac \subset M_k^2$, is also tangent to M_k^2. Hence the directions u and v of the segments ca and cb at c and the vector $-n_c$ lie in the same 2-dimensional direction at c. Therefore the angles between the three vectors satisfy

$$\angle(-n_c, u) + \angle(u, v) = \angle(-n_c, v).$$

The last angle, being the angle between the interior normal $-n_c$ and the chord cb of the definitely convex body C, is $\leq \pi/2$. The first angle $\angle(-n_c, u) = \alpha$ since n_c is antipodal to n_a. Thus

$$\alpha + \gamma \leq \pi/2. \tag{15}$$

Take a point c' on the diameter ab such that the lengths $ac' = ac = t$. In the case $k = -1$, out of the triangle acc', one has

$$\cosh t \cdot \tan(\alpha/2) = \cot \gamma' \quad \text{where} \quad \gamma' = \angle acc'. \tag{16}$$

Obviously $\gamma' \leq \gamma$. Now by (15),

$$\gamma' \leq \pi/2 - \alpha. \tag{17}$$

Taking cotangent of both parts and combining the result with (16), one has

$$\cosh t \cdot \tan(\alpha/2) \geq \tan\gamma; \tag{18}$$

$$\cosh t \geq 1 + \sec\alpha > 2 \quad \text{for} \quad \alpha \in (0, \pi/2). \tag{19}$$

by (14),

$$\cosh d \geq \cosh t > 2 \tag{20}$$

which contradicts (11).

For the case $k = 1$, replace cosh by cos everywhere up to (19) inclusive. Then (19) will be a contradiction. The case $k = 0$ is trivial.

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