TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1994

HADAMARD PRODUCT OF CERTAIN MEROMORPHIC STARLIKE AND CONVEX FUNCTIONS

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Abstract. The author establishes certain results concerning the Hadamard product of meromorphic starlike and meromorphic convex functions analogous to those obtained by Vinod Kumar (J. Math. Anal. Appl. 113(1986), 230-234).

1. Introduction

Throughout the paper, let the functions of the form

$$\phi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0, c_n \geq 0),$$

and

$$\psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \qquad (d_1 > 0, d_n \geq 0)$$

be regular and univalent in the unit disc $U = \{z : |z| < 1\}$; and let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0, a_n \geq 0),$$

$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \quad (a_{0,i} > 0, a_{n,i} \geq 0),$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (b_0 > 0, b_n \geq 0),$$

Received January 8, 1993.
1991 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Hadamard product, meromorphic starlike functions, meromorphic convex functions, positive coefficients.

and

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{0,j} > 0, b_{n,j} \geq 0)$$

be regular and univalent in the punctured disc $D = \{z : 0 < |z| < 1\}$.

Let $S_0^*(\alpha, \beta)$ denote the class of functions $\phi(z)$ which satisfy the condition

$$|z\phi'(z)/\phi(z) - 1| < \beta|z\phi'(z)/\phi(z) + 1 - 2\alpha|$$

for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all $z \in U$; and let $C_0^*(\alpha, \beta)$ be the class of functions $\phi(z)$ for which $z\phi'(z) \in S_0^*(\alpha, \beta)$. It is well known that the functions in $S_0^*(\alpha, \beta)$ and $C_0^*(\alpha, \beta)$ are respectively starlike and convex of order α and type β with negative coefficients in U.

Denote by $\sum S_0^*(\alpha, \beta)$, the class of functions $f(z)$ which satisfy the condition

$$|zf'(z)/f(z) + 1| < \beta|zf'(z)/f(z) + 2\alpha - 1|$$

for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all $z \in D$; and let $\sum C_0^*(\alpha, \beta)$ be the class of functions $f(z)$ for which $zf'(z) \in \sum S_0^*(\alpha, \beta)$. The functions in $\sum S_0^*(\alpha, \beta)$ and $\sum C_0^*(\alpha, \beta)$ are respectively called meromorphic starlike and meromorphic convex of order α and type β with positive coefficients in D. The class $\sum S_0^*(\alpha, \beta)$ with $a_0 = 1$ has extensively been studied in [4].

Using similar arguments as given in [4], we can easily prove the following results for functions in $\sum S_0^*(\alpha, \beta)$ and $\sum C_0^*(\alpha, \beta)$.

A function $f(z) \in \sum S_0^*(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} [\{n + 1 + \beta(n + 2\alpha - 1)\} a_n] \leq 2\beta(1 - \alpha) a_0;$$

and $f(z) \in \sum C_0^*(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} [n\{n + 1 + \beta(n + 2\alpha - 1)\} a_n] \leq 2\beta(1 - \alpha) a_0.$$

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa [5,6,7], Kumar [1,2,3], and others. Accordingly, the quasi-Hadamard product of two functions $\phi(z)$ and $\psi(z)$ is defined by

$$\phi * \psi(z) = c_1 d_1 z - \sum_{n=2}^{\infty} c_n d_n z^n.$$

Let us define the Hadamard product of two meromorphic univalent functions $f(z)$ and $g(z)$ by

$$f * g(z) = \frac{a_0 b_0}{z} + \sum_{n=1}^{\infty} a_n b_n z^n.$$

The Hadamard product of more than two meromorphic univalent functions can similarly be defined.

In [2], Kumar obtained certain results concerning the quasi-Hadamard product of two or more functions in $S_0^*(\alpha, \beta)$ and $C_0^*(\alpha, \beta)$, and has shown that his results improve the corresponding results of Owa [7].

Since to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, one is tempted to search results analogous to Kumar [2] for meromorphic univalent functions in D. Thus we introduce the following class of meromorphic univalent functions in D.

A function $f(z) \in \sum_k^*(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} \left[n^k\{n+1+\beta(n+2\alpha-1)\}a_n\right] \leq 2\beta(1-\alpha)a_0, \tag{1}$$

where $0 \leq \alpha < 1, 0 < \beta \leq 1$ and k is any fixed nonnegative real number.

Evidently, $\sum_0^*(\alpha, \beta) \equiv \sum S_0^*(\alpha, \beta)$ and $\sum_1^*(\alpha, \beta) \equiv \sum C_0^*(\alpha, \beta)$. Further, $\sum_k^*(\alpha, \beta) \subset \sum_h^*(\alpha, \beta)$ if $k > h \geq 0$, the containment being proper. Moreover, for any positive integer k, we have the following inclusive relation.

$$\textstyle\sum_k^*(\alpha, \beta) \subset \sum_{k-1}^*(\alpha, \beta) \subset \cdots \subset \sum_2^*(\alpha, \beta) \subset \sum C_0^*(\alpha, \beta) \subset \sum S_0^*(\alpha, \beta).$$

We also note that for every nonnegative real number k, the class $\sum_k^*(\alpha, \beta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} n^{-k} \left\{\frac{2\beta(1-\alpha)}{n+1+\beta(n+2\alpha-1)}\right\} a_0 \lambda_n z^n,$$

where $a_0 > 0, \lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n \leq 1$, satisfy the inequality (1).

The aim of the present paper is to establish certain results concerning the Hadamard product of meromorphic starlike and convex functions analogous to those obtained by Kumar [2]. It is interesting to note that our results are valid for the usual "Hadamard product" while Kumar [2] used in his results "quasi-Hadamard product" instead of the usual "Hadamard product".

2. The Main Theorems

Theorem 1. *Let the functions $f_i(z)$ belong to the class $\sum C_0^*(\alpha, \beta)$ for every $i = 1, 2, \cdots, m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{2m-1}^*(\alpha, \beta)$.*

Proof. it is sufficient to show that

$$\sum_{n=1}^{\infty} \left[n^{2m-1}\{n+1+\beta(n+2\alpha-1)\} \prod_{i=1}^{m} a_{n,i}\right] \leq 2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i}\right].$$

Since $f_i(z) \in \sum C_0^*(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}[n\{n+1+\beta(n+2\alpha-1)\}a_{n,i}] \leq 2\beta(1-\alpha)a_{0,i}, \tag{2}$$

for every $i=1,2,\cdots,m$. Therefore,

$$n\{n+1+\beta(n+2\alpha-1)\}a_{n,i} \leq 2\beta(1-\alpha)a_{0,i}$$

or

$$a_{n,i} \leq \left[\frac{2\beta(1-\alpha)}{n\{n+1+\beta(n+2\alpha-1)\}}\right]a_{0,i},$$

for every $i=1,2,\cdots,m$. The right-hand expression of the last inequality is not greater than $n^{-2}a_{0,i}$. Hence

$$a_{n,i} \leq n^{-2}a_{0,i}, \tag{3}$$

for every $i=1,2,\cdots,m$.

Using (3) for $i=1,2,\cdots,m-1$, and (2) for $i=m$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[n^{2m-1}\{n+1+\beta(n+2\alpha-1)\}\prod_{i=1}^{m}a_{n,i}\right]\\
\leq&\sum_{n=1}^{\infty}\left[n^{2m-1}\{n+1+\beta(n+2\alpha-1)\}\left(n^{-2(m-1)}\prod_{i=1}^{m-1}a_{0,i}\right)a_{n,m}\right]\\
=&\left[\prod_{i=1}^{m-1}a_{0,i}\right]\sum_{n=1}^{\infty}[n\{n+1+\beta(n+2\alpha-1)\}a_{n,m}]\\
\leq&2\beta(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\right].
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \sum_{2m-1}^*(\alpha,\beta)$.

Theorem 2. *Let the functions $f_i(z)$ belong to the class $\sum S_0^*(\alpha,\beta)$ for every $i=1,2,\cdots,m$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $\sum_{m-1}^*(\alpha,\beta)$.*

Proof. Since $f_i(z) \in \sum S_0^*(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}[\{n+1+\beta(n+2\alpha-1)\}a_{n,i}] \leq 2\beta(1-\alpha)a_{0,i}, \tag{4}$$

for every $i=1,2,\cdots,m$. Therefore

$$a_{n,i} \leq \left[\frac{2\beta(1-\alpha)}{\{n+1+\beta(n+2\alpha-1)\}}\right]a_{0,i},$$

and hence

$$a_{n,i} \leq n^{-1} a_{0,i}, \tag{5}$$

for every $i = 1, 2, \cdots, m$.

Using [5] for $i = 1, 2, \cdots, m-1$, and [4] for $i = m$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[n^{m-1}\{n+1+\beta(n+2\alpha-1)\} \prod_{i=1}^{m} a_{n,i} \right] \\
\leq &\sum_{n=1}^{\infty} \left[n^{m-1}\{n+1+\beta(n+2\alpha-1)\} \left(n^{-(m-1)} \prod_{i=1}^{m-1} a_{0,i} \right) a_{n,m} \right] \\
= &\left[\prod_{i=1}^{m-1} a_{0,i} \right] \sum_{n=1}^{\infty} [\{n+1+\beta(n+2\alpha-1)\} a_{n,m}] \\
\leq &2\beta(1-\alpha) \left[\prod_{i=1}^{m} a_{0,i} \right].
\end{aligned}$$

Hence $f_1 * f_2 * \cdots * f_m(z) \in \sum_{m-1}^{*}(\alpha, \beta)$.

Theorem 3. *Let the functions $f_i(z)$ belong to the class $\sum C_0^*(\alpha, \beta)$ for every $i = 1, 2, \cdots, m$; and let the functions $g_j(z)$ belong to the class $\sum S_0^*(\alpha, \beta)$ for every $j = 1, 2, \cdots, q$. Then the Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $\sum_{2m+q-1}^{*}(\alpha, \beta)$.*

Proof. It is sufficient to show that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left[n^{2m+q-1}\{n+1+\beta(n+2\alpha-1)\} \left(\prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right) \right] \\
\leq &2\beta(1-\alpha) \left(\prod_{i=1}^{m} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right).
\end{aligned}$$

Since $f_i(z) \in \sum C_0^*(\alpha, \beta)$, the inequalities [2] and [3] hold for every $i = 1, 2, \cdots, m$. Further, since $g_j(z) \in \sum S_0^*(\alpha, \beta)$, we have

$$\sum_{n=1}^{\infty} [\{n+1+\beta(n+2\alpha-1)\} b_{n,j}] \leq 2\beta(1-\alpha) b_{0,j}, \tag{6}$$

for every $j = 1, 2, \cdots, q$. Whence we obtain

$$b_{n,j} \leq n^{-1} b_{0,j}, \tag{7}$$

for every $j = 1, 2, \cdots, q$

Using [3] for $i=1,2,\cdots,m$; [7] for $j=1,2,\cdots,q-1$; and [6] for $j=q$, we get

$$\sum_{n=1}^{\infty}\left[n^{2m+q-1}\{n+1+\beta(n+2\alpha-1)\}\left(\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right)\right]$$

$$\leq\sum_{n=1}^{\infty}\left[n^{2m+q-1}\{n+1+\beta(n+2\alpha-1)\}\left(n^{-2m}n^{-(q-1)}\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right)b_{n,q}\right]$$

$$=\left(\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right)\sum_{n=1}^{\infty}[\{n+1+\beta(n+2\alpha-1)\}b_{n,q}]$$

$$\leq 2\beta(1-\alpha)\left(\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,j}\right)$$

Hence $f_1*f_2*\cdots*f_m*g_1*g_2*\cdots*g_q\in\sum_{2m+q-1}^{*}(\alpha,\beta)$.

We note that the required estimate can also be obtained by using [3] for $i=1,2,\cdots,m-1$; [7] for $j=1,2,\cdots,q$; and [2] for $i=m$.

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