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A CONDITION FOR SIMPLE RING IMPLYING FIELD II

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Abstract. It is shown that if R is a simple ring with identity 1 and with a nonzero idempotent e and satisfies the condition $(P_2)_e$:

$(P_2)_e$ If $e-(a_1b_1+a_2b_2)$ is a right (left) zero divisor in R, then so is $e-(b_1a_1+b_2a_2)$.

then R is a field.Thus if R is a simple ring then eRe is a field for every nonzero idempotent e in R if it exists and eRe satisfies $(P_2)_e$. We also discuss the above property for the simple ring case by eliminating the identity 1.

1. Introduction

Throughout the note R will denote an associative ring. An element a in R is called a right (left) zero divisor ([2], p. 88) if there exists a nonzero element b of R such that $ba=0$ ($ab=0$); a is a zero divisor if it is a right and a left zero divisor. Clearly 0 is a zero divisor if R has more than one element.

Let R be a ring with a nonzero idempotent e, i.e., $e^2=e$, and let n be a positive integer. We define the property $(P_n)_e$ as follows:

$(P_n)_e$ If $e-\sum_{i=1}^n a_ib_i$ is a right (left) zero divisor in R, then so is $e-\sum_{i=1}^n b_ia_i$.

Let R satisfy $(P_n)_e$. Then R satisfies $(P_m)_e$ for every $m=1,2,\cdots,n-1$. It is easy to show that if f is an isomorphism from R onto a ring S then $f(e)^2=f(e^2)=f(e)$ and S satisfies the property $(P_n)_{f(e)}$. R is called simple ([2], p. 107) if $R^2=R$ and the only ideals in R are 0 and R. Note that if R is a simple ring then eRe is also a simple ring with identity e.

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R is called semiprime if the only ideal of R which squares to zero is the zero ideal. Note that if R is a semiprime ring then $Rx \neq 0$ and $xR \neq 0$ for every nonzero element x in R. We show that if R is a semiprime ring then every right (left) identity of R is an identity.

Let R have identity 1. We shall denote the property $(P_n)_1$ by (P_n). It is easy to prove that R always satisfies (P_1). As the proof of lemma 1 of [4], we have that R satisfies (P_n) for all positive integers n if and only if R satisfies (P_2). Noncommutative rings satisfying (P_2) actually exist, e.g., the Examples 1 and 2 of [3].

The purpose of this note is to prove the following theorem: if R is a simple ring with identity 1 and with a nonzero idempotent e and satisfies $(P_2)_e$ then R is a field. In particular, any simple ring with identity 1 satisfying (P_2) must be a field. Thus we obtain the following result: if R is a simple ring then eRe is a field for every nonzero idempotent e in R if it exists and eRe satisfies $(P_2)_e$. We ask whether the hypothesis "R has identity 1" can be eliminated in the above theorem.

Let R be a ring. We shall denote the Jacobson radical by $J(R)$. R is called local if R has the unique maximal ideal. We also discuss the above property for the simple ring case by eliminating the identity 1. For the related results, see [5] – [8].

2. Results

We begin with two easy known results.

Proposition 1. *If R is a semiprime ring, then every right (left) identity of R is an identity.*

Proof. Let e be a right identity of R. Assume that x is a nonzero element in R. Since R is semiprime, $Rx \neq 0$. Thus, we obtain $R(ex) = (Re)x = Rx \neq 0$. Hence, $ex \neq 0$. Thus, $e(ex - x) = 0$ implies $ex = x$. So, e is an identity of R. The proof of the other (left) case is similar.

Proposition 2. *If R is a ring with identity 1, then R satisfies (P_1).*

Proof. Assume that $1 - ab$ is not a right zero divisor in R. Let c be an element of R such that $c(1 - ba) = 0$. Then $c = cba$ and $cb(1 - ab) = c(1 - ba)b = 0$. By assumption, $cb(1 - ab) = 0$ implies $cb = 0$. Thus, $c = cba = 0$. Hence $1 - ba$ is not a right zero divisor. The proof of the other (left) case is similar.

As the proof of Lemma 1 of [4], we have the

Lemma 1. *If R is a ring with identity 1, then R satisfies (P_n) for all positive integers n if and only if R satisfies (P_2).*

Lemma 2. *If R is a ring with identity 1 and satisfies (P_2), then every commutator $xy - yx$ is a zero divisor for all x, y in R.*

Proof. Since R satisfies (P_2), by Lemma 1, R satisfies (P_n) for all positive integers

n. For all x, y in R, we have $1-(1-(xy-xy))=0$. Thus by $(P_3), xy-yx = 1-(1-(xy-yx))$ is a zero divisor, as desired.

Lemma 3. *If R is a simple ring with identity 1 and satisfies (P_2), then R is a field.*

Proof. Let c be a nonzero element of R. We show that c is neither a right nor a left zero divisor. Since R is simple, $RcR = R$. Thus, there exist a_i, b_i in $R, i = 1, 2, \cdots, m$, such that $\sum_{i=1}^{m} a_i c b_i = 1$. Let $d = \sum_{i=1}^{m} b_i a_i$. By Lemma 1, R satisfies (P_n) for all positive integers n.
Hence using this twice, $1 = 1-(1-\sum_{i=1}^{m} a_i c b_i)$ implies $cd = 1-(1-cd)$ and $dc = 1-(1-dc)$ are neither right nor left zero divisors. It follows from these that c is neither a right nor a left zero divisor.

By Lemma 2, $xy-yx$ is a zero divisor for all x, y in R. Thus by the result above, $xy = yx$. Hence R is a commutative simple ring with identity. So, R is a field, as desired.

Using Lemma 3, we obtain the

Corollary. *If R is a simple ring, then eRe is a field for every nonzero idempotent e in R if it exists and eRe satisfies $(P_2)_e$.*

Theorem 1. *If R is a simple ring with identity 1 and with a nonzero idempotent e and satisfies $(P_2)_e$, then R is a field.*

Proof. Since R is a simple ring, eRe is a simple ring with identity e and satisfies $(P_2)_e$. Thus by Lemma 3, eRe is a field. Hence we obtain $exeye = (exe)(eye) = (eye)(exe) = eyexe$ for all x, y in R. Using ([1], p. 24, Corollary), R is isomorphic to the ring of all $m \times m$ matrices over a field F for some positive integer m. Let $f : R \to F_m$ be this isomorphism. Suppose $m > 1$. Because of $f(e)^2 = f(e)$, by the well known result of linearr algebra there exists an invertible matrix A in F_m such that $Af(e)A^{-1} = \sum_{i=1}^{k} e_{ii}$ for some positive integer k. Let $B = \sum_{i=1}^{k} e_{ii}$. Therefore we have that $R \cong F_m \cong AF_mA^{-1} = F_m$. So, F_m satisfies the property $(P_2)_B$. Thus by Lemma 1, F_m satisfies $(P_n)_B$ for all positive integers n. If $k = m$, then $f(e) = A^{-1}BA = 1$ and so $e = 1$. Hence by Lemma 3, R is a field, so we may assume $k < m$. Then $B - \sum_{i=k+1}^{m}(-e_{1i})e_{i1}$ is a zero divisor in F_m . Since F_m satisfies $(P_{m-k})_{B'}$ this implies $1 = B - \sum_{i=k+1}^{m} e_{i1}(-e_{1i})$ is a zero divisor, a contradiction. Thus $m = 1$. So, $R \cong F$. Hence R is a field.

In Theorem 1, if we eliminate the identity 1 then we obtain the

Theorem 2.*If R is a simple ring with a nonzero idempotent e and satisfies $(P_2)_e$, then either R is a field; or eR and Re are local rings which consist of zero divisors and $eR/J(eR)$ and $Re/J(Re)$ are fields.*

Proof. Using Theorem 1, eRe is a field. By symmetry, we only consider the right ideal eR of R. Clearly, eR has a left identity e. Assume that A is a nonzero right ideal of eR and $ae \neq 0$ for some a in A. Then there exists an element c of R such that $aece = (eae)(ece) = e$, since eRe is a field. Thus, $e = aece \in A$ and so $A = eR$. Hence, either eR is a simple ring or $Ae = 0$ for every proper right ideal A of eR.

Assume that eR is a simple ring. By Proposition 1, e is the identity of eR. Thus, we have

$$(*) \qquad exe = (ex)e = e(ex) = ex \text{ for all } x \text{ in } R.$$

Let $B = \{b \in R : eb = 0\}$. Obviously, B is a right ideal of R. Assume that $b \in B$ and $x \in R$. Then by $(*)$, we obtain $exb = (exe)b = ex(eb) = 0$. Hence B is an ideal of R. By the simplicity of R and $eR \neq 0$, we get $B = 0$. Thus, $e(xe - exe) = 0$ implies $xe = exe$. By $(*)$, $ex = exe = xe$. Hence e is central. So, $eR = R$. Therefore, $R = eR = eRe$ is a field.

Assume that $Ae = 0$ for every proper right ideal A of eR. Let I be the sum of all proper right ideals of eR. Then I is the unique maximal right ideal in R. Thus $I = J(eR)$. Hence eR is a local ring. For all x in R, since $xe = e-(e-xe)$ is a right zero divisor, by $(P_2)_e$, $ex = e - (e - ex)$ is also a right zero divisor. Clearly, ex is a left zero divisor. Thus eR consists of zero divisors. As above, we can easily show that $eR/J(eR)$ is a field.

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