TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 2, Summer 1994

EFFECTIVE ESTIMATES FOR BOUNDARY VALUE PROBLEMS

GOU-SHENG YANG AND JYH-WEN LEU

Abstract. We establish some effective estimates for the boundary value problems

$$x^{(6)}(t)-\sum_{i=0}^{5}p_i(t)x^{(i)}(t)=0,\quad x(a)=x'(a)=x''(a)=x'''(a)=x^{(4)}(a)=x(b)=0;$$

and

$$x^{(6)}(t)-\sum_{i=0}^{5}p_i(t)x^{(i)}(t)=0,\quad x(a)=x'(a)=x''(a)=x(b)=x'(b)=x''(b)=0.$$

Introduction

Bogar and Gustafson ([1]) have shown that the homogeneous bundary value problem

$$x^{(6)}-\sum_{i=0}^{2}p_i(t)x^{(i)}=0; \tag{1}$$

$$x(a)=x'(a)=x''(a)=x'''(a)=x(b)=x'(b)=0; \tag{2}$$

where $p_i\in C[a,b]$, $0\le i\le 2$, has only the trivial solution provided the inequality

$$\frac{377.4}{10^6}(b-a)^6\|p_0\|+\frac{156.91}{10^5}(b-a)^5\|p_1\|+\frac{9}{2048}(b-a)^4\|p_2\|<1; \tag{3}$$

is satisfied, where $\|p_i\|=\sup(|p_i(t)|,t\in[a,b])$. For the complete differential equation

$$x^{(6)}-\sum_{i=0}^{5}p_i(t)x^{(i)}=0,; \tag{4}$$

Received June 30, 1993, revised October 27, 1993.
1991 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Boundary value problem, trivial solution.

where $p_i \in C[a,b]$, $0 \le i \le 5$, Agarwal and Milovanovic ([2]) have shown that the boundary value problem (4),(2) has only the trivial solution provided the inequality

$$\frac{1}{32805}(b-a)^6\|p_0\| + \frac{25+34\sqrt{10}}{911250}(b-a)^5\|p_1\| + \frac{1}{360}(b-a)^4\|p_2\| + \frac{1}{30}(b-a)^3\|p_3\| + \frac{1}{5}(b-a)^2\|P_4\| + \frac{2}{3}(b-a)\|p_5\| = \theta < 1 \tag{5}$$

is satisfied.

The main purpose of this paper is to obtain a similar result of Milovanovic in considering the boundary conditions

$$x(a) = x'(a) = x''(a) = x'''(a) = x^{(4)}(a) = x(b) = 0; \tag{6}$$

$$x(a) = x'(a) = x''(a) = x(b) = x'(b) = x''(b) = 0; \tag{7}$$

respectively, instead of the boundary conditions(2).

Main result

Theorem A. *The boundary value problem (4),(6) has only the trivial solution provided the inequality*

$$\frac{625}{6718464}(b-a)^6\|p_0\| + \frac{1}{720}(b-a)^5\|p_1| + \frac{1}{72}(b-a)^4\|p_2\| + \frac{1}{12}(b-a)^3\|p_3\| + \frac{1}{3}(b-a)^2\|p_4\| + \frac{5}{6}(b-a)\|p_5\| = \theta_1 < 1; \tag{8}$$

is satisfied.

Theorem B. *The boundary value problem (4),(7) has only the trivial solution provided the inequality*

$$\frac{1}{46080}(b-a)^6\|p_p\| + \frac{159+76\sqrt{6}}{3000000}(b-a)^5\|p_1\| + \frac{1}{480}(b-a)^4\|p_2\| + \frac{41+28\sqrt{7}}{9720}(b-a)^3\|p_3\| + \frac{13}{120}(b-a)^2\|p_4\| + \frac{1}{2}(b-a)\|p_5\| < 1; \tag{9}$$

is satisfied.

For the proof of the theorems, we need the following:

Lemma 1. *Any function $x \in C^{(6)}[0,1]$ satisfying the conditions*

$$x(0) = x'(0) = x''(0) = x'''(0) = x^{(4)}(0) = x(1) = 0; \tag{10}$$

can be written as

$$x(t) = t^4(1-t)F(t) \tag{11}$$

where

$$\begin{aligned} F(t) &= \int_0^t \frac{F_1(t_1)}{t_1^2}dt_1, & F_1(t) &= \int_0^t \frac{F_2(t_2)}{t_2^2}dt_2, \\ F_2(t) &= \int_0^t \frac{F_3(t_3)}{t_3^2}dt_3, & F_3(t) &= \int_0^t \frac{F_4(t_4)}{t_4^2}dt_4, \\ F_4(t) &= \int_1^t \frac{t_5^4}{(1-t_5)^6}F_5(t_5)dt_5, & F_5(t) &= \int_1^t (1-t_6)^5 x^{(6)}(t_6)dt_6. \end{aligned} \tag{12}$$

proof.

Let $\phi(t)$ be the right hand side of (11) and satisfying the condition (10). Then

$$\phi(t) = t^4(1-t)F(t)$$

$$\phi'(t) = [4t^3(1-t) - t^4]F(t) + t^2(1-t)F_1(t) \tag{13}$$

$$\phi''(t) = [12t^2(1-t) - 8t^3]F(t) + [6t(1-t) - 2t^2]F_1(t) + (1-t)F_2(t) \tag{14}$$

$$\begin{aligned} \phi'''(t) = {} & [24t(1-t) - 36t^3]F(t) + [18(1-t) - 18t]F_1(t) \\ & + [\frac{6(1-t)}{t} - 3]F_2(t) + [\frac{1-t}{t^2}]F_3(t) \end{aligned} \tag{15}$$

$$\begin{aligned} \phi^{(4)}(t) = {} & [24(1-t) - 96t]F(t) + [\frac{24(1-t)}{t} - 72]F_1(t) \\ & + [\frac{12(1-t)}{t^2} - \frac{24}{t}]F_2(t) + [\frac{4(1-t)}{t^3} - \frac{4}{t_2}]F_3(t) \\ & + [\frac{1-t}{t_4}]F_4(t) \end{aligned} \tag{16}$$

$$\begin{aligned} \phi^{(5)}(t) = {} & [-120]F(t) + [-120t^{-1}]F_1(t) + [-60t^{-2}]F_2(t) \\ & + [-20t^{-3}]F_3(t) + [-5t^{-4}]F_4(t) + [(1-t)^{-5}]F_5(t) \end{aligned} \tag{17}$$

$$\begin{aligned} \phi^{(6)}(t) = {} & [\frac{-120}{t^2}]F_1(t) + [\frac{120}{t^2}]F_1(t) + [\frac{-120}{t^3}]F_2(t) + [\frac{120}{t^3}]F_2(t) \\ & + [\frac{-60}{t^4}]F_3(t) + [\frac{60}{t^4}]F_3(t) + [\frac{-20}{t^5}]F_4(t) + [\frac{20}{t^5}]F_4(t) \\ & + [\frac{-5}{(1-6)^6}]F_5(t) + [\frac{5}{(1-t)^6}]F_5(t) + [x^{(6)}(t)] = x^{(6)}(t) \end{aligned}$$

It follows that

$$\phi(t) = x(t) + \frac{1}{120}c_1t^5 + \frac{1}{24}c_2t^4 + \frac{1}{6}c_3t^3 + \frac{1}{2}c_4t^2 + c_5t + c_6$$

for some constants c_i, $1 \le i \le 6$.
Since $\phi(t)$ satisfies (10), it follows that

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0,$$

hence $\phi(t) = x(t)$.

Lemma 2. *Let x be as in Lemma 1. Then*

$$\begin{aligned} |x(t)| &\le \frac{625}{6718464}M \\ |x'(t)| &\le \frac{1}{720}M \\ |x''(t)| &\le \frac{1}{72}M \\ |x'''(t)| &\le \frac{1}{12}M \\ |x^{(4)}(t)| &\le \frac{1}{3}M \\ |x^{(5)}(t)| &\le \frac{5}{6}M \end{aligned}$$

where $M = \max_{0\le t\le 1}|x^{(6)}(t)|$

Proof. From(12), it is immediately that

$$\begin{array}{llll} |F_5(t)| \le & \frac{1}{6}M(1-t)^6, & |F_4(t)| \le & \frac{1}{30}Mt^5, \\ |F_3(t)| \le & \frac{1}{120}Mt^4, & |F_2(t)| \le & \frac{1}{360}Mt^3, \\ |F_1(t)| \le & \frac{1}{720}Mt^2, & |F(t)| \le & \frac{1}{720}Mt. \end{array} \tag{18}$$

Thus, it follows from (11) and (18) that

$$|x(t)| \le \frac{1}{720}t^5(1-t)M. \tag{19}$$

The right hand side of (19) attains its maximum at $t = \frac{2}{3}$, so that

$$|x(t)| \le \frac{625}{6718464}M, \qquad 0 \le t \le 1.$$

It follows from (13) and (18) that

$$\begin{aligned} |x'(t)| &\le |4t^3(1-t) - t^4|\frac{t}{720}M + |t^2(1-t)|\frac{t^2}{720}M \\ &= \frac{M}{720}t^4[(1-t) + |4-5t|] \\ &= \frac{M}{720}\begin{cases} t^4(5-6t), & 0 \le t \le \frac{4}{5}, \\ t^4(4t-3), & \frac{4}{5} \le t \le 1. \end{cases} \end{aligned} \tag{20}$$

The right hand side of (20) attains its maximum at $t = 1$, so that

$$|x'(t)| \le \frac{M}{720}, \qquad 0 \le t \le 1.$$

From (14) and (18), we have

$$
\begin{aligned}
|x''(t)| \leq & |12t^2(1-t) - 8t^3| \frac{t}{720} M + |6t(1-t) - 2t^2| \frac{t^2}{720} M + |(1-t)| \frac{t^3}{360} M \\
= & \frac{M}{360} t^3 [|6 - 10t| + |3 - 4t| + (1-t)] \\
= & \frac{M}{360} \begin{cases} t^3(10 - 15t), & 0 \leq t \leq \frac{3}{5}, \\ t^3(-2 + 5t), & \frac{3}{5} \leq t \leq \frac{3}{4}, \\ t^3(-8 + 13t), & \frac{3}{4} \leq t \leq 1. \end{cases}
\end{aligned} \tag{21}
$$

The right hand side of (21) attains its maximum at $t = 1$, so that

$$
|x''(t)| \leq \frac{M}{72}, \qquad 0 \leq t \leq 1.
$$

From (15) and (18), we have

$$
\begin{aligned}
|x'''(t)| \leq & |24t(1-t) - 36t^2| \frac{t}{720} M + |18(1-t) - 18t| \frac{t^2}{720} M \\
& + |\frac{6(1-t)}{t} - 3| \frac{t^3}{360} M + |\frac{1-t}{t^2}| \frac{t^4}{120} M \\
= & \frac{M}{120} t^2 [2|2 - 5t| + 3|1 - 2t| + |2 - 3t| + (1-t)] \\
= & \frac{M}{120} \begin{cases} t^2(10 - 20t), & 0 \leq t \leq \frac{2}{5}, \\ 2t^2, & \frac{2}{5} \leq t \leq \frac{1}{2}, \\ t^2(-8 + 12t), & \frac{1}{2} \leq t \leq \frac{2}{3}, \\ t^2(-8 + 18t), & \frac{2}{3} \leq t \leq 1. \end{cases}
\end{aligned} \tag{22}
$$

The right hand side of (22) attains its maximum at $t = 1$, so that

$$
|x'''(t)| \leq \frac{M}{12}, \qquad 0 \leq t \leq 1.
$$

From (16) and (18), we have

$$
\begin{aligned}
|x^{(4)}(t)| \leq & |24t(1-t) - 96t| \frac{t}{720} M + |\frac{24(1-t)}{t} - 72| \frac{t^2}{720} M + |\frac{12(1-t)}{t^2} - \frac{24}{t}| \frac{t^3}{360} M \\
& + |\frac{4(1-t)}{t^3} - \frac{4}{t^2}| \frac{t^4}{120} M + |\frac{1-t}{t^4}| \frac{t^5}{30} M \\
= & \frac{M}{30} t [|1 - 5t| + |1 - 4t| + |1 - 3t| + |1 - 2t| + (1-t)] \\
= & \frac{M}{30} \begin{cases} t(5 - 15t), & 0 \leq t \leq \frac{1}{5}, \\ t(3 - 5t), & \frac{1}{5} \leq t \leq \frac{1}{4}, \\ t(1 + 3t), & \frac{1}{4} \leq t \leq \frac{1}{3}, \\ t(-1 + 9t), & \frac{1}{3} \leq t \leq \frac{1}{2} \\ t(-3 + 13t), & \frac{1}{2} \leq t \leq 1. \end{cases}
\end{aligned} \tag{23}
$$

The right hand side of (23) attains its maximum at $t = 1$,so that

$$|x^{(4)}(t)| \leq \frac{M}{3}, \qquad 0 \leq t \leq 1.$$

Finally, from (17) and (18), we have

$$\begin{aligned}|x^{(5)}(t)| &\leq 120\frac{t}{720}M + \frac{120}{t}\frac{t^2}{720}M + F\frac{60}{t^2}\frac{t^3}{360}M + \frac{20}{t^3}\frac{t^4}{120}M \\ &\quad + \frac{5}{t^4}\frac{t^5}{30}M + \frac{1}{(1-t)^5}\frac{1}{6}(1-t)^6 M \\ &= \frac{M}{6}(1+4t), \quad 0 \leq t \leq 1. \end{aligned} \tag{24}$$

The right hand side of (24) attains its maximum at $t = 1$, so that

$$|x^{(5)}(t)| \leq \frac{5}{6M}, \qquad 0 \leq t \leq 1.$$

This completes the proof of Lemma 2.

Lemma 3. *Let $x \in C^{(6)}[a,b]$, and satisfy the conditions (6). Then*

$$\begin{aligned} |x(t)| &\leq \frac{625}{6718464}(b-a)^6\mu, \\ |x'(t)| &\leq \frac{1}{720}(b-a)^5\mu, \\ |x''(t)| &\leq \frac{1}{72}(b-a)^4\mu, \\ |x'''(t)| &\leq \frac{1}{12}(b-a)^3\mu, \\ |x^{(4)}(t)| &\leq \frac{1}{3}(b-a)^2\mu, \\ |x^{(5)}(t)| &\leq \frac{5}{6}(b-a)\mu, \end{aligned}$$

where $\mu = \max_{a\leq t\leq b}|x^{(6)}(t)|$. The inequalities are the best possible as the identity holds for the function $x(t) = (t-a)^5(b-t)$.

Proof. The proof requires only the transformation $\mu = a + (b-a)t$, $0 \leq t \leq 1$, in Lemma 2.

Lemma 4. *Any function $x \in C^{(6)}[0,1]$ satisfying the conditions*

$$x(0) = x'(0) = x''(0) = x(1) = x'(1) = x''(1) = 0 \tag{25}$$

can be written as

$$x(t) = t^2(1-t)^3G(t), \tag{26}$$

where

$$G(t) = \int_0^t t_1^{-2} G_1(t_1)dt_1, \qquad G_1(t) = \int_0^t t_2^{-2} G_2(t_2)dt_2,$$

$$G_2(t) = \int_1^t t_3^2(1-t_3)^{-4} G_3(t_3)dt_3, \quad G_3(t) = \int_1^t (1-t_4)^{-2} G_4(t_4)dt_4, \tag{27}$$

$$G_4(t) = \int_1^t (1-t_5)^{-2} G_5(t_5)dt_5, \qquad G_5(t) = \int_1^t (1-t_6)^5 x^{(6)}(t_6)dt_6,$$

proof. Let $\psi(t)$ be the right hand side of (26) and satisfying the condition (25). Then

$$\psi(t) = t^2(1-t)^3 G(t)$$

$$\psi'(t) = [2t(1-t)^3 - 3t^2(1-t)^2]G(t) + [(1-t)^3]G_1(t) \tag{28}$$

$$\psi''(t) = [2(1-t)^3 - 12t(1-t)^2 + 6t^2(1-t)]G(t) + [\frac{2(1-t)^3}{t} - 6(1-t)^2]G_1(t) + [\frac{(1-t)^3}{t}]G_2(t) \tag{29}$$

$$\psi'''(t) = [-18(1-t)^2 + 36t(1-t) - 6t^2]G(t) + [\frac{-18(1-t)^2}{t} + 18(1-t)]G_1(t) + [\frac{-9(1-t)^2}{t^2}]G_2(t) + [\frac{1}{1-t}]G_3(t) \tag{30}$$

$$\psi^{(4)}(t) = [72(1-t) - 48t]G(t)] + [\frac{72(1-t)}{t} - 24]G_1(t) + [\frac{36(1-t)}{t^2}]G_2(t) + [\frac{-8}{(1-t)^2}]G_3(t) + [\frac{1}{(1-t)^3}]G_4(t) \tag{31}$$

$$\psi^{(5)}(t) = [-120]G(t) + [\frac{-120}{t}]G_1(t) + [\frac{-60}{t^2}]G_2(t) + [\frac{20}{(1-t)^3}]G_3(t) + [\frac{-5}{(1-t)^4}]G_4(t) + [\frac{1}{(1-t)^5}]G_5(t) \tag{32}$$

$$\begin{aligned}\psi^{(6)}(t) &= [\frac{-120}{t^2}]G_1(t) + [\frac{120}{t^2}]G_1(t) + [\frac{-120}{t^3}]G_2(t) + [\frac{120}{t^3}]G_2(t) \\ &\quad + [\frac{-60}{(1-t)^4}]G_3(t) + [\frac{60}{(1-t)^4}]G_3(t) + [\frac{-20}{(1-t)^5}]G_4(t) + [\frac{20}{(1-t)^5}]G_4(t) \\ &\quad + [\frac{-5}{(1-t)^6}]G_5(t) + [\frac{5}{(1-t)^6}]G_5(t) + x^{(6)}(t) \\ &= x^{(6)}(t).\end{aligned}$$

It follows that

$$\psi(t) = x(t) + \frac{1}{120}c_1t^5 + \frac{1}{24}c_2t^4 + \frac{1}{6}c_3t^3 + \frac{1}{2}c_4t^2 + c_5t + c_6$$

for some constants c_i, $1 \le i \le 6$.

Since $\psi(t)$ satisfies (25), we find that

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0.$$

Hence $\psi(t) = x(t)$.

Lemma 5. *Let x be as in Lemma 4. Then*

$$|x(t)| \le \frac{1}{46080}M,$$
$$|x'(t)| \le \frac{159+76\sqrt{6}}{3000000}M,$$
$$|x''(t)| \le \frac{1}{480}M,$$
$$|x'''(t)| \le \frac{41+28\sqrt{7}}{9720}M,$$
$$|x^{(4)}(t)| \le \frac{13}{120}M,$$
$$|x^{(5)}(t)| \le \frac{1}{2}M,$$

where $M = \max_{0\le t\le 1}|x^{(6)}(t)|$.

proof. From (27), it is immediately that

$$\begin{aligned} &|G_5(t)| \le \tfrac{1}{6}(1-t)^6 M, && |G_4(t)| \le \tfrac{1}{30}(1-t)^5 M,\\ &|G_3(t)| \le \tfrac{1}{120}(1-t)^4 M, && |G_2(t)| \le \tfrac{1}{360}t^3 M, \\ &|G_1(t)| \le \tfrac{1}{720}t^2 M, && |G(t)| \le \tfrac{1}{720}tM. \end{aligned} \tag{33}$$

Thus, it follows from (26) and (33) that

$$|x(t)| \le \frac{1}{720}t^3(1-t)^3 M. \tag{34}$$

The right hand side of (34) attains its maximum at $t = \frac{1}{2}$, so that

$$|x(t)| \le \frac{1}{46080}M.$$

It follows from (28) and (33) that

$$\begin{aligned} |x'(t)| \le& |2t(1-t)^3 - 3t^2(1-t)|\frac{t}{720}M + |(1-t)^3|\frac{t^2}{720}M \\ =& \frac{M}{720}t^2(1-t)^2[|2-5t| + (1-t)] \\ =& \frac{M}{720}\begin{cases} t^2(1-t)^2(3-6t), & 0 \le t \le \frac{2}{5}, \\ t^2(1-t)^2(-1+4t), & \frac{2}{5} \le t \le 1. \end{cases} \end{aligned} \tag{35}$$

The right hand side of (35) attains its maximum at $t = \frac{4+\sqrt{6}}{10}$, so that

$$|x'(t)| \leq \frac{159 + 76\sqrt{6}}{3000000} M, \qquad 0 \leq t \leq 1.$$

From (29) and (33), we have

$$\begin{aligned}
|x''(t)| &\leq |2(1-t)^3 - 12t(1-t)^2 + 6t^2(1-t)| \frac{t}{720} M \\
&\quad + |\frac{2(1-t)^3}{t} - 6(1-t)| \frac{t^2}{720} M + |\frac{(1-t)^3}{t^2}| \frac{t^3}{360} M \\
&= \frac{M}{360} t(1-t)[|10t^2 - 8t + 1| + |4t^2 - 5t + 1| + (t^2 - 2t + 1)] \\
&= \frac{M}{360} \begin{cases} t(1-t)(15t^2 - 15t + 3), & 0 \leq t \leq \frac{4-\sqrt{6}}{10}, \\ t(1-t)(-5t^2 + t + 1), & \frac{4-\sqrt{6}}{10} \leq t \leq \frac{1}{4}, \\ t(1-t)(-13t^2 + 11t - 1), & \frac{1}{4} \leq t \leq \frac{4+\sqrt{6}}{10}, \\ t(1-t)(7t^2 - 5t + 1), & \frac{4+\sqrt{6}}{10} \leq t \leq 1, \end{cases} \\
&\leq \frac{M}{360} \begin{cases} \frac{1}{4}(15t^2 - 15t + 3), & 0 \leq t \leq \frac{4-\sqrt{6}}{10}, \\ \frac{1}{4}(-5t^2 + t + 1), & \frac{4-\sqrt{6}}{10} \leq t \leq \frac{1}{4}, \\ \frac{1}{4}(-13t^2 + 11t - 1), & \frac{1}{4} \leq t \leq \frac{4+\sqrt{6}}{10}, \\ \frac{1}{4}(7t^2 - 5t + 1), & \frac{4+\sqrt{6}}{10} \leq t \leq 1. \end{cases} \qquad (36)
\end{aligned}$$

The right hand side of (36) attains its maximum at $t = 0$ and $t = 1$, so that

$$|x''(t)| \leq \frac{M}{480}, \qquad 0 \leq t \leq 1.$$

From (30) and (33), we have

$$\begin{aligned}
|x'''(t)| &\leq | - 18(1-t)^2 + 36t(1-t) - 6t^2| \frac{t}{720} M + |\frac{-18(1-t)^2}{t} + 18(1-t)| \frac{t^2}{720} M \\
&\quad + |\frac{-9(1-t)^2}{t^2}| \frac{t^3}{360} M + |\frac{1}{(1-t)}| \frac{(1-t)^4}{120} M \\
&= \frac{M}{120}[t| - 3 + 12t - 10t^2| + t| - 3 + 9t - 6t^2| + 3t(1-t)^2 + (1-t)^3] \\
&= \frac{M}{120} \begin{cases} (18t^3 - 24t^2 + 6t + 1), & 0 \leq t \leq \frac{6-\sqrt{6}}{10}, \\ (-2t^3 + 1), & \frac{6-\sqrt{6}}{10} \leq t \leq \frac{1}{2}, \\ (-14t^3 + 18t^2 - 6t + 1), & \frac{1}{2} \leq t \leq \frac{6+\sqrt{6}}{10}, \\ (6t^3 - 6t^2 + 1), & \frac{6+\sqrt{6}}{10} \leq t \leq 1. \end{cases} \qquad (37)
\end{aligned}$$

The right hand side of (37) attains its maximum at $t = \frac{4-\sqrt{7}}{9}$, so that

$$|x'''(t)| \leq \frac{41 + 28\sqrt{7}}{9720}, \qquad 0 \leq t \leq 1.$$

From (31) and (33), we have

$$\begin{aligned}
|x^{(4)}(t)| &\le |72(1-t)-48t|\frac{t}{720}M+|\frac{72(1-t)}{t}-24|\frac{t^2}{720}M+|\frac{36(1-t)}{t^2}|\frac{t^3}{360}M \\
&\quad +|\frac{-8}{(1-t)^2}|\frac{(1-t)^4}{120}M+|\frac{1}{(1-t)^3}|\frac{(1-t)^5}{30}M \\
&= \frac{M}{30}[t|3-5t|+t|3-4t|+3t(1-t)+3(1-t)^2] \\
&= \frac{M}{30}\begin{cases}(3+3t-9t^2), & 0\le t\le \frac{3}{5}, \\ (3-3t+t^2), & \frac{3}{5}\le t\le \frac{3}{4}, \\ (3-9t+9t^2), & \frac{3}{4}\le t\le 1.\end{cases}
\end{aligned} \tag{38}$$

The right hand side of (38) attains its maximum at $t=\frac{1}{6}$, so that

$$|x^{(4)}(t)| \le \frac{13}{120}, \qquad 0\le t\le 1.$$

Finally, from (32) and (33), we have

$$\begin{aligned}
|x^{(5)}(t)| &\le |-120|\frac{t}{720}M+|\frac{-120}{t}|\frac{t^2}{720}M+|\frac{-60}{t^2}|\frac{t^3}{360}M \\
&\quad +|\frac{20}{(1-t)^3}|\frac{(1-t)^4}{120}M+|\frac{-5}{(1-t)^4}|\frac{(1-t)^5}{30}M+|\frac{1}{(1-t)^5}|\frac{(1-t)^6}{6}M \\
&= \frac{t}{2}M+\frac{1}{2}(1-t)M \\
&= \frac{1}{2}M, \quad 0\le t\le 1.
\end{aligned}$$

This completes the proof of this Lemma.

Lemma 6. *Let $x\in C^{(6)}[a,b]$, and satisfy the conditions (7). Then*

$$\begin{aligned}
|x(t)| &\le \frac{1}{46080}(b-a)^6\mu, \\
|x'(t)| &\le \frac{159+76\sqrt{6}}{3000000}(b-a)^5\mu, \\
|x''(t)| &\le \frac{1}{480}(b-a)^4\mu, \\
|x'''(t)| &\le \frac{41+28\sqrt{7}}{9720}(b-a)^3\mu, \\
|x^{(4)}(t)| &\le \frac{13}{120}(b-a)^2\mu, \\
|x^{(5)}(t)| &\le \frac{1}{2}(b-a)\mu,
\end{aligned}$$

where $\mu = \max_{a\le t\le b} |x^{(6)}(t)|$. *The inequalities are the best possible as the identity holds for the function* $x(t) = (t-a)^3(b-t)^3$.

proof. The proof require only the transformation $\mu = a + (b-a)t$, $0 \le t \le 1$, in Lemma 5.

Proof of Theorem A

Suppose on the contrary that the boundary value problem (4) and (6) has a nontrivial solution $x(t)$. Then $\mu = \max_{a\le t\le b} |x^{(6)}(t)| \ne 0$, otherwise $x(t)$ would be a polynomial of degree $m < 6$ on $[a,b]$, and $x^{(m)}(t)$ would not vanish on $[a,b]$ which cannot satisfy the boundary conditions (6).

Thus, if $\mu = |x^{(6)}(t_1)|$ for some t_1 in $[a,b]$, then from the differential equation (4), we have

$$\begin{aligned}\mu =& |x^{(6)}(t_1)| = |\sum_{i=0}^{5} p_i(t_1)x^{(i)}(t_1)| \\ \le& \sum_{i=0}^{5} \|p_i\| |x^{(i)}(t_1)|.\end{aligned}$$

Now, use Lemma 3 in the above inequality, we have

$$\begin{aligned}\mu \le& \frac{625}{6718464}(b-a)^6\|p_0\|\mu + \frac{1}{720}(b-a)^5\|p_1\|\mu \\ &+ \frac{1}{72}(b-a)^4\|p_2\|\mu + \frac{1}{12}(b-a)^3\|p_3\|\mu \\ &+ \frac{1}{3}(b-a)^2\|p_4\|\mu + \frac{5}{6}(b-a)\|p_5\|\mu \\ =& \theta\mu.\end{aligned}$$

We note that at least one of the numbers $\|p_i\|$, $0 \le i \le 5$, is different from zero, otherwise, again $x(t)$ would be a polynomial of degree less than 6, and cannot satisfy the boundary conditions (6). Hence, it is necessary that $\theta \ge 1$.
This completes the proof of theorem A.

The proof of theorem B is similar to that of Theorem A. We omit the detail.

References

1. G.A. Bogar and G.B. Gustafson, "Effective estimates of invertibility intervals for linear multipoint boundary value problems," *J. Differential Equations* 29(1978), 180-204.
2. R.P. Agarwal, G.V. Milovanovic, "On an inequality of Bogar and Gustafson," *J. math. anal. appl.* 146, 207-216 (1990).

Department of Mathematics, Tamkang University, Tamsui, Taiwan, 25137.