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ON IDEALS OF THE COEFFICIENT RINGS IN GROUP RINGS

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Abstract. Let R and S be rings, G any group. If the group rings RG and SG are isomorphic as rings, we formulate a correspondence between the ideals of R and those of S and show that this correspondence is one-to-one in case R and S are isomorphic. It is shown that this correspondence also works for Jordan ideals, provided that G is abelian.

1. Introduction

In the study of group rings, one comes across two types of isomorphism problems. The first isomorphism problem asks as to what extent the group ring RG determines the group G. The answer in general is no. The second isomorphism problem asks whether or not R is an invariant of RG. More precisely if for any two rings R and S and a fixed group G, RG isomorphic to SG as a ring, does it follow that R is isomorphic to S? The answer is again no in general. While studying the first isomorphism problem, a normal subgroup correspondence has been established (e.g. see [1], [4], [7], [8]). We take up the case when $RG \cong SG$ and formulate a correspondence between the ideals (i.e. two sided ideals) of R and S. We also show that this correspondence is one-to-one in case the second isomorphism problem is resolved. Finally we show that this correspondence also works for Jordan ideals (Lie ideals), provided that G is an abelian group.

2. Ideal-Correspondence

Proposition 2.1. *Let R and S be rings, G any group and $\theta : RG \cong SG$ a ring isomorphism. For every ideal I of R, set*

$$\phi(I) = \{s \in S | s1_G \in \theta(IG)\}$$

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Then
(i) *$\phi(I)$ is an ideal of S and for $I = R$, $\phi(I) = S$*
Thus we have a correspondence

$$\phi : \mathcal{L}_g(R) \longrightarrow \mathcal{L}_g(S)$$

where $\mathcal{L}_g(R)$ and $\mathcal{L}_g(S)$ denote lattices of ideals of R and S respectively.
Similarly we have a correspondence

$$\psi : \mathcal{L}_g(S) \longrightarrow \mathcal{L}_g(R)$$

(ii) *If θ induces an isomorphism between R and S, we have*

$$\phi\psi = 1, \qquad \psi\phi = 1$$

i.e. ϕ is a one-to-one correspondence.

Proof. (i) For any ideal I of R, IG is an ideal of RG and as θ is a ring homomorphism of RG onto SG, $\theta(IG)$ is an ideal of SG.

$$\because O_S 1_G \in \theta(IG) \Longrightarrow O = O_S \in \phi(I)$$

$$\therefore \phi(I) \text{ is a non-empty subset of } S.$$

Now take any $s, t \in \phi(I)$. Then $s1_G \in \theta(IG)$, $t1_G \in \theta(IG)$ Since $\theta(IG)$ is an ideal of SG, $s1_G - t1_G \in \theta(IG)$, so $(s-t)1_G \in \theta(IG)$ and hence $s - t \in \phi(I)$.

This proves that $\phi(I)$ is an additive subgroup of S. Now take any $t \in \phi(I)$ and $s \in S$. We again use the fact that $\theta(IG)$ is an ideal of SG to get $(t1_G)(s1_G) \in \theta(IG)$ and $(s1_G)(t1_G) \in \theta(IG)$.

$$\because (ts)1_G \in \theta(IG) \text{ and } (st)1_G \in \theta(IG)$$

$$\Longrightarrow ts \in \phi(I) \text{ and } st \in \phi(I).$$

$$\therefore \phi(I) \text{ is an ideal of } S.$$

Clearly for $I = R, \phi(I) = S$.
(ii) Since $\phi(I)$ is an ideal of S,

$$\therefore \phi(I)G \text{ is an ideal of } SG.$$

Take any element $\sum s_g g \in \phi(I).\ G,\ s_g \in \phi(I) \forall g \in G$.

Then

$$\begin{aligned}\theta^{-1}(\sum s_g g) &= \theta^{-1}(\sum (s_g 1_G)(1_s\, g)) \\ &= \sum \theta^{-1}[(s_g 1_G)(1_s\, g)] \\ &= \sum \theta^{-1}(s_g 1_G)\theta^{-1}(1_s\, g)\end{aligned}$$

$$\because s_g \in \phi(I), \quad \therefore s_g 1_G \in \theta(IG).$$

This implies that $\theta^{-1}(s_g 1_G) \in IG$.

Also $\theta^{-1}(1_s\, g) \in RG$ and since IG is an ideal of RG

$$\begin{aligned}&\therefore \theta^{-1}(s_g 1_G)\theta^{-1}(1_s g) \in IG \\ &\Longrightarrow \sum \theta^{-1}(s_g 1_G)\theta^{-1}(1_{-s} g) \in IG \\ &\Longrightarrow \theta^{-1}(\sum s_g g) \in IG.\end{aligned}$$

Thus we see that

$$\theta^{-1}(\phi(I)G) \subseteq IG \tag{1}$$

Consider $\psi : \mathcal{L}_g(S) \longrightarrow \mathcal{L}_g(R)$. Then

$\psi(J) = \{rR | r1_G \in \theta^{-1}(JG)\}$ for every ideal J of S.

Claim $\phi\psi = 1$, $\psi\phi = 1$

For any ideal I of R, consider $\psi\phi(I)$. Take any $r \in \psi\phi(I)$. Then $r \in R$ and $r1_G \in \theta^{-1}(\phi(I)G)$ i.e. $r \in R$ and $r1_G \in IG$ (using (1)), hence $r \in I$.

$\therefore \psi\phi(I) \subseteq I$.

Conversely take any $r \in I$. Then $r \in \psi\phi(I)$ iff $r1_G \in \theta^{-1}(\phi(I)G)$ i.e. iff $\theta(r1_G) \in \phi(I)G$. Now as θ induces an isomorphism between R and S,

$\therefore \theta(r1_G) = s1_G$ for some $s \in S$, and so $s \in \phi(I)$ $\quad (\because s1_G \in \theta(IG))$.

$\therefore \theta(r1_G) \in \phi(I)G$

$\Longrightarrow r.1_G \in \theta^{-1}(\phi(I)G)$

$\Longrightarrow r \in \psi\phi(I)$.

$\therefore I \subseteq \psi\phi(I)$

$\therefore \psi\phi = 1$

Similarly $\phi\psi = 1$.

Hence ϕ is a one-to-one correspondence provided that θ induces an isomorphism between R and S.

Remark 2.2. There are many instances when θ does induce an isomorphism between R and S. We mention a few:

(i) If R and S are P.I.D's which do not contain fields and $< x >$ is an infinite cyclic group then $R < x > \cong S < x >$, implies that $R \cong S$([5]).

(ii) If R and S are integral domains and G is a torsion abelian group then $RG \cong SG$ implies that $R \cong S$([6]).

Example 2.3. Now we give an example to show that in case $\theta : RG \cong SG$ does not induce an isomorphism from R to S, the correspondence obtained by us need not be a one-to-one correspondence.

Let G_0, G_1, $G_2, \ldots$ be infinite cyclic groups and $R = Z$, $S = ZG_0$ and $G = \prod_{i \geq 1} G_i$, the direct product of $G_1, G_2, \ldots$ Then we have an isomorphism $\theta : RG \cong SG$ which does not induce an isomorphism between R and S.

Take $J = \triangle_Z(G_0)$, the augmentation ideal of ZG_0. J is an ideal of S but there does not exist any ideal I of R such that $J = \phi(I)$.

Thus the correspondence ϕ is not a bijective one.

Now we study a few properties of the correspondence ϕ (irrespective of the fact that it is bijective or not).

Proposition 2.4. (i) If I is a nilpotent ideal of R, then $\phi(I)$ is a nilpotent ideal of S.

(ii) Suppose R is a commutative ring without proper zero divisors and I is a torsion ideal of R, then $\phi(I)$ is a torsion ideal of S.

(iii) Suppose G is an abelian group, R a commutative ring and I is a nil-ideal of R, then $\phi(I)$ is a nil-ideal of S.

Proof. (i) Since I is a nilpotent ideal of R, $I^n = 0$ for some integer $n > 1$.

Now $(IG)^n \subseteq I^n G = 0 = 0_R 1_G$ It follows that $(\theta(IG))^n = 0$. Take any $s_1, s_2, \ldots, s_n \in \phi(I)$. Then $s_1, s_2, \ldots, s_n \in S$ and $s_1 1_G, s_2 1_G, \ldots, s_n 1_G \in \theta(IG)$. Since $(\theta(IG))^n = 0$, $s_1 1_G \cdot s_2 1_G \cdots s_n 1_G = 0$. This implies $(s_1 s_2 \cdots s_n) 1_G = 0 1_G$, hence $s_1 s_2 \cdots s_n = 0$

Thus $(\phi(I))^n = 0$ and so $\phi(I)$ is a nilpotent ideal of S.

(ii) Take any $r_1 g_1 + r_2 g_2 + \ldots + r_n g_n \in IG$, $r_1, r_2, \ldots r_n \in I$; $g_1, g_2, \ldots, g_n \in G$. Since I is a torsion ideal of R, there exist $t_1, t_2, \ldots, t_n \in R$ such that $t_i r_i = 0$ $(i = 1, 2, \ldots, n, t_i \neq 0)$. Then $t = t_1 t_2 \cdots t_n \neq 0$ (as R is without zero divisors) and $(t 1_G)(r_1 g_1 + r_2 g_2 + \ldots + r_n g_n) = 0$, so IG is a torsion ideal of RG.

Now take any $\theta(\xi) \in \theta(IG)$, $\xi \in IG$. Since IG is a torsion ideal of RG, there exists $\eta \in RG$, $\eta \neq 0$ such that $\eta\xi = 0$. Then $\theta(\eta\xi) = 0$ i.e. $\theta(\eta)\theta(\xi) = 0$. As θ is $1 - 1$ and $\eta \neq 0$, $\theta(\eta) \neq 0$ and so we see that $\theta(IG)$ is a torsion ideal of SG.

Finally take any $s \in \phi(I)$. Then $s \in S$ and $s 1_G \in \theta(IG)$. Since $\theta(IG)$ is a torsion ideal of SG, there exists $s_1 g_1 + s_2 g_2 + \ldots + s_n g_n \neq 0 \in SG$ ($g_1, g_2, \ldots, g_n$ distinct elements of G) such that $(s_1 g_1 + s_2 g_2 + \ldots + s_n g_n)(s 1_G) = 0$. This implies that $s_1 g_1 + s_2 g_2 + \ldots + s_n g_n = 0$.

$$\therefore s_1 s = 0, \ s_2 s = 0, \ldots, s_n s = 0$$

$$\because s_1 g_1 + s_2 g_2 + \ldots + s_n g_n \neq 0, \ \therefore s_j \neq 0 \ \text{ for some } j (1 \leq j \leq n).$$

$\therefore$ We see that for every $s \in \phi(I)$, there exists $s_j \in S$, $s_j \neq 0$ such that $s_j s = 0$. Hence s is a torsion element of S.

$\therefore \phi(I)$ is a torsion ideal of S.

(iii) Since I is a nil-ideal of R and RG is commutative, IG is a nil-ideal of RG. Take any $\theta(\xi) \in \theta(IG)$; $\xi \in IG$. Since IG is a nil-ideal of RG, there exists an integer $n > 1$ such that $\xi^n = 0$. Now $(\theta(\xi))^n = \theta(\xi^n) = \theta(0) = 0 \therefore \theta(IG)$ is a nil-ideal of SG. Take any $s \in \phi(I)$. Then $s \in S$ such that $s1_G \in \theta(IG)$. Since $\theta(IG)$ is a nil-ideal of SG, there exists an integer $n > 1$ such that $(s1_G)^n = 0_S1_G$ i.e. $s^n1_G = 0_S1_G$ and so we must have $s^n = 0$. Thus s is nilpotent and $\phi(I)$ is shown to be a nil-ideal of S.

3. Jordan ideal Correspondence

In what follows, we assume that G is an abelian group. We show that given any ring isomorphism $\theta : RG \cong SG$ of group rings, to every Jordan ideal (Lie ideal) of R, there corresponds a Jordan ideal (Lie ideal) of S.

Definition 3.1. ([2],[3]) An additive subgroup U of a ring R is said to be a Jordan ideal of R if whenever $u \in U$ and $r \in R$, then $u \circ r = ur + ru$ is in U.

Definition 3.2. ([2],[3]) An additive subgroup U of a ring R is said to be a Lie ideal of R if whenever $u \in U$ and $r \in R$, then $[u, r] = ur - ru$ is in U.

Remark 3.3. ([2]) Every two sided ideal is a Jordan (Lie) ideal but converse is not true in general.

Now we proceed to show that if $\theta : RG \cong SG$ and I is a Jordan ideal of R, then $\phi(I)$ is a Jordan ideal of S, where $\phi(I) = \{s \in S | s1_G \in \theta(IG)\}$.

If I is a Jordan ideal of R, it is not difficult to show that IG is a Jordan ideal of RG.

Take any arbitrary element $\xi \in \theta(IG)$ and $\eta \in SG$. Then $\xi = \theta(\alpha)$ for some $\alpha \in IG$. Also as θ is on-to, there exists $\beta \in RG$ such that $\theta(\beta) = \eta$
Then

$$\begin{aligned} &\xi \circ \eta = \xi\eta + \eta\xi \\ &=\theta(\alpha)\theta(\beta) + \theta(\beta)\theta(\alpha) \\ &=\theta(\alpha\beta) + \theta(\beta\alpha) \\ &=\theta(\alpha\beta + \beta\alpha) \in \theta(IG) \qquad (\alpha\beta + \beta\alpha \in IG \text{ as } IG \text{ is a Jordan ideal of } RG) \end{aligned}$$

$\therefore \theta(IG)$ is a Jordan ideal of SG.

Now to see that $\phi(I)$ is a Jordan ideal of S, it suffices to show that for $t \in \phi(I)$ and $s \in S$, $ts + st \in \phi(I)$.

$$t \in \phi(I) \Longrightarrow t \in S \text{ and } t1_G \in \theta(IG).$$

Also $s1_G \in SG$ and since $\theta(IG)$ is a Jordan ideal of SG,

$$\begin{aligned} & (t1_G)(s1_G) + (s1_G)(t1_G) \in \theta(IG) \\ \Longrightarrow\ & (ts1_G) + (st1_G) \in \theta(IG) \\ \Longrightarrow\ & (ts + st)1_G \in \theta(IG) \\ \Longrightarrow\ & ts + st \in \phi(I) \end{aligned}$$

$\therefore \phi(I)$ is a Jordan ideal of S.

Essentially the same argument as above gives that if I is a Lie ideal of R, then $\phi(I)$ is Lie ideal of S.

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