TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 3, Autumn 1994

LINEAR AUTONOMOUS NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN THE PHASE SPACE OF REGULATED FUNCTIONS

LUIZ FICHMANN

Abstract. We extend the natural description of the spectrum for the flow of the linear equation $\frac{d}{dt}Dx_t = Lx_t$ from the context of continuous functions to the context of regulated right-continuous functions.

1. Introduction

This paper intends to present a result contained in [3], namely, the extension of the spectral results of [6] for the flow of the Linear Autonomous NFDEs to the context of regulated right-continuous functions.

If $[a,b]$ is an interval of the real line and X is a Banach space, we write $G([a,b],X)$ for the space of the functions $\psi : [a,b] \to X$ for which there exist the limits $\psi(t^+)$ for every $t \in [a,b[$ and $\psi(t^-)$ for every $t \in]a,b]$. Such functions are called regulated functions.

In [3] we extend some results obtained by J. Hale ([4]) and D. Henry ([6]) for the so-called Neutral Functional Differential Equations (NFDEs), which have the form $\frac{d}{dt}(x(t)-f(t,x_t)) = g(t,x_t)$, from the context of continuous functions to the context of regulated functions. The motivation for this extension is the fact that the fundamental matrix, which appears in the variation-of-constants formula of the linear non-homogeneous NFDE ([4], [6]), is regulated and not continuous in t. So, the space of regulated functions appears as a natural context to include the fundamental matrix or the resolvent, in the case we consider a generic Banach space X. In this general context, Hönig ([8],[9]) studied the Volterra-Stieltjes linear Integral Equations. We applied these results, since the initial value problem of a linear NFDE leads to such an integral equation ([2]).

Another extension of the phase-space C of continuos functions, for which the variation -of-constants formula has a functional analytic sense, was done by Diekmann in [1] for retarded equations (NFDEs with $f \equiv 0$), where the author developed the theory called

Received August 5, 1992; revised April 6, 1994.
1991 *Mathematics Subject Classification.* 34K40.
Key words and phrases. Neutral functional equation, spectrum, regulated functions.

"sun-star calculus". This theory consists of considering the adjoint of the flow of the linear equation in the dual space and taking the restriction to the maximal subspace where strong-continuity holds and again taking the adjoint. In this way C is embedded into the product space $M_\infty = \mathbb{R}^n \times L_\infty$, a nice space to include the fundamental matrix, and one can use the variation-of-constants formula in the weak-* sense.

Another possible approach is to choose $M_p = \mathbb{R}^n \times L_p$ as the phase-space. This was done in [10] for a certain class of NFDEs.

2. The main result

Let $\mathbb{E}^n$ denote the Euclidean space of real or complex n-vectors and let r be a fixed positive number. $\mathcal{G}^+ = G^+([-r,0],\mathbb{E}^n)$ is the space of the regulated right-continuous functions $\varphi : [-r,0] \to \mathbb{E}^n$, which is complete with the norm $\|\varphi\| = \sup_{-r\leq\theta\leq 0} \|\varphi(\theta)\|$. We call $C = C([-r,0],\mathbb{E}^n)$ the closed subspace of $\mathcal{G}^+$ of continuous functions. If x is a regulated right-continuous map of $[a-r,b]$ into $\mathbb{E}^n$, then $x_t \in \mathcal{G}^+$ is given, for each $a \leq t \leq b$, by $x_t(\theta) = x(t+\theta), -r \leq \theta \leq 0$.

Let D, L be fixed continuous linear functionals from $\mathcal{G}^+$ into $\mathbb{E}^n$, with integral representations given by $D\varphi = \varphi(0) - \int_{-r}^{0} d\mu(\theta)\varphi(\theta)$ and $L\varphi = \int_{-r}^{0} d\eta(\theta)\varphi(\theta)$ for $\varphi \in \mathcal{G}^+$; where μ, η are matrix-valued functions (from $[-r,0]$ into $\mathcal{L}(\mathbb{E}^n)$) of bounded variation which vanish at $\theta = 0$ and are left-continuous. For these represrentations, we utilize the Interior Integral which extends the Riemann-Stieltjes Integral (see [8]). We assume here that μ has no singular part, i.e., $\int_{-r}^{0} d\mu(\theta)\varphi(\theta) = \sum_{k=1}^{\infty} A_k\varphi(-r_k) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta$, $\forall\varphi \in \mathcal{G}^+$, where $0 \leq r_k \leq r$ and $A_k \in \mathcal{L}(\mathbb{E}^n)$ for $k \in \mathrm{N}$ and $A \in L_1([-r,0],\mathcal{L}(\mathbb{E}^n))$.

In this situation, the initial value problem is well posed for the NFDE:

$$\frac{d}{dt}Dx_t = Lx_t, \qquad t \geq 0 \tag{N}$$

that is, for $\varphi \in \mathcal{G}^+$ we have the unique regulated right-continuous solution $x = x(0,\varphi)$ of (N) for $t \geq 0$ with $x_0 = \varphi$. We have, then, well defined the flow of $(N), \{T(t)\}_{t\geq 0}$, semigroup of bounded linear operators on $\mathcal{G}^+$ given by $T(t)\varphi = x_t(0,\varphi)$ for $\varphi \in \mathcal{G}^+$ and $t \geq 0$.

Let D^0 be the jump part of D, that is, $D^0\varphi = \varphi(0) - \sum_{k=1}^{\infty} A_k\varphi(-r_k)$ for $\varphi \in \mathcal{G}^+$. We denote by $\mathcal{G}^+_{D^0}$ the kernel of D^0. The initial value problem is also well posed for the difference equation $(D)_0 : D^0x_t = 0,\ t \geq 0$. This defines the flow of $(D)_0, \{T^0(t)\}_{t\geq 0}$, semigroup of bounded linear operators on $\mathcal{G}^+_{D^0}$.

We known that C is invariant under $T(t)$ $(t \geq 0)$ and $C_{D^0} \overset{def.}{=} \mathcal{G}^+_{D^0} \cap C$ is invariant under $T^0(t)$ $(t \geq 0)$, that is, the solution of (N) or $(D)_0$ is continuous whenever the initial data is a continuous function.

Daniel Henry ([6], [7]) gives a complete description of the spectrum of the operators $T^0(t)_{|C_{D^0}}$ and $T(t)_{|C}$ for $t \geq 0$, using the infinitesimal generator $\mathbf{A}^0$ of $\{T^0(t)_{|C_{D^0}}\}_{t\geq 0}$

and $\mathbb{A}$ of $\{T(t)_{|\mathcal{C}}\}_{t\geq 0}$. The restriction of each flow, as above, is a strongly continuous semigroup of linear operators which admits a closed infinitesimal generator with dense domain in $\mathcal{C}_{D^0}$ and $\mathcal{C}$, respectively. Namely $\mathcal{D}(\mathbf{A}^0) = \{\varphi \in \mathcal{C}_{D^0} \mid \varphi' \in \mathcal{C}_{D^0}\}$ with $\mathbf{A}^0\varphi = \varphi'$ for $\varphi \in \mathcal{D}(\mathbf{A}^0)$, and $\mathcal{D}(\mathbb{A}) = \{\varphi \in \mathcal{C} \mid \varphi' \in \mathcal{C}$ and $D\varphi' = L\varphi\}$ with $\mathbb{A}\varphi = \varphi'$ for $\varphi \in \mathcal{D}(\mathbb{A})$. For the spectrum of these generators, we have:

$$\sigma(\mathbf{A}^0) = P\sigma(\mathbf{A}^0) = \{\lambda \in \mathbb{C} \mid \det H(\lambda) = 0\}$$
$$\sigma(\mathbb{A}) = P\sigma(\mathbb{A}) = \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}$$

where $\det H(\lambda) = 0$ and $\det \Delta(\lambda) = 0$ are the respective characteristic equations of $(D)_0$ and (N), i.e.:

$$H(\lambda) = I - \sum_{k=1}^{\infty} A_k e^{-\lambda r_k} = D^0(e^{\lambda \cdot} I)$$

and $\Delta(\lambda) = \lambda H(\lambda) - \lambda \int_{-r}^{0} A(\theta)e^{\lambda\theta}d\theta - \int_{-r}^{0} d\eta(\theta)e^{\lambda\theta} = \lambda D(e^{\lambda \cdot} I) - L(e^{\lambda \cdot} I)$. Henry shows that:

$$\sigma(T^0(t)\,|_{\mathcal{C}_{D^0}})\backslash\{0\} = \overline{e^{t\sigma(\mathbf{A}^0)}}\backslash\{0\} \quad a.e. \quad \text{in } t \geq 0$$

and $T(t)_{|\mathcal{C}} - T^0(t) \circ \Psi_{|\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a compact operator for each $t \geq 0$, (where the map Ψ above is a continuous projection from $\mathcal{G}^+$ onto $\mathcal{G}^+_{D^0}$ such that $\Psi(\mathcal{C}) \subset \mathcal{C}_{D^0}$, defined in the next section), and with these facts he concludes that:

$$\sigma(T(t)_{|\mathcal{C}})\backslash\{0\} = \overline{e^{t\sigma(\mathbb{A})}}\backslash\{0\} \quad a.e. \quad \text{in } t \geq 0.$$

The flows of $(D)_0$ and (N) are neither strongly continuous nor something like "strongly regulated", for if we have a jump $T(t)\varphi(\theta) - T(t)\varphi(\theta^-) = 2l$, with $\|l\| > 0$, for some $t \geq 0$ and $\theta \in]-r, 0]$, then $\|T(t+\epsilon_1)\varphi - T(t+\epsilon_2)\varphi\| > \|l\|$ for any $\epsilon_1 \neq \epsilon_2$ in $]0, \delta[$, for some small $\delta > 0$. Then, we cannot extend the infinitesimal generators to dense domains in $\mathcal{G}^+_{D^0}$ and $\mathcal{G}^+$ respectively. Nevertheless, we still can show that the results obtained by Henry are extensible for $\mathcal{G}^+_{D^0}$ and $\mathcal{G}^+$ respectively. This is done in the next sections.

3. The difference equation

Let $\mathcal{E}^+ \subset \mathcal{G}^+$ be the space of step-functions, that is:

$$\mathcal{E}^+ = \{\varphi \in \mathcal{G}^+ | \varphi = \sum_{i=1}^{k} c_i \chi_{[\theta_i, 0]} \text{ for some } k \in \mathbb{N}^*, c_i \in \mathbb{E}^n \text{ and } -r \leq \theta_i \leq 0, i = 1, 2, \cdots, k,\}$$

where, for $J \subset [-r, 0], \chi_J(\theta) = 1$ if $\theta \in J$ and $\chi_J(\theta) = 0$ if $\theta \notin J$.

$\mathcal{E}^+$ is a dense subspace of $\mathcal{G}^+$ (see [8]).

Let $G^-BV_0 = G^-BV_0([-r, 0], (\mathbb{E}^n)')$ be the space of applications $\alpha : [-r, 0] \to (\mathbb{E}^n)' = \mathcal{L}(\mathbb{E}^n, \mathbb{E})$ with bounded variation which vanish at $\theta = 0$ and are left-continuous.

We have the following immediate lemmas:

Lemma 1. *For each $\varphi \in \mathcal{E}^+$, there is a sequence $\{\varphi_m\}_{m\in\mathbb{N}}$, $\varphi_m \in \mathcal{C}$, such that $\varphi_m(0) = \varphi(0)$, $\|\varphi_m\| = \|\varphi\|$ $\forall m \in \mathbb{N}$ and*

$$\int_{-r}^{0} d\alpha(\theta)\varphi_m(\theta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\theta)\varphi(\theta) \qquad \forall \alpha \in G^-BV_0.$$

Proof. 1) Suppose first $n = 1$ and $\varphi = \chi_{[\theta,0]} \in \mathcal{E}^+, -r \leq \theta \leq 0$. If $\theta = -r$ take $\varphi_m = \varphi \equiv 1$ and if $\theta \neq -r$, for $m > \frac{1}{r+\theta}$ take

$$\varphi_m(\beta) = \chi^{(m)}_{[\theta,0]}(\beta) \overset{def.}{=} \begin{cases} 0 & \text{if } \ -r \leq \beta \leq \theta - \frac{1}{m} \\ m(\beta - \theta + \frac{1}{m}) & \text{if } \ \theta - \frac{1}{m} \leq \beta \leq \theta \\ 1 & \text{if } \ \theta \leq \beta \leq 0. \end{cases}$$

Then, for $\alpha \in G^-BV_0$ we have

$$\int_{-r}^{0} d\alpha(\beta)\chi^{(m)}_{[\theta,0]}(\beta) = \int_{\theta-\frac{1}{m}}^{\theta} d\alpha(\beta)[m(\beta-\theta)] - \alpha(\theta - \frac{1}{m})$$

$$\overset{m\to\infty}{\longrightarrow} -\alpha(\theta) = \int_{-r}^{0} d\alpha(\beta)\chi_{[\theta,0]}(\beta),$$

$$\text{since } \left|\int_{\theta-\frac{1}{m}}^{\theta} d\alpha(\beta)[m(\beta-\theta)]\right| \leq Var_{[\theta-\frac{1}{m},\theta]}[\alpha] \overset{m\to\infty}{\longrightarrow} 0.$$

2) For the general case, we remember first that, for $\alpha \in G^-BV_0$, we have the scalar functions $\alpha_j \in G^-BV_0([-r,0],\mathbb{E}), j = 1,2,\cdots,n$, such that for each $p = (p_1,\cdots,p_n) \in \mathbb{E}^n$ and $\theta \in [-r,0]$ we have $\alpha(\theta)p = \sum_{j=1}^{n} \alpha_j(\theta)p_j$, and for $\varphi \in \mathcal{G}^+, \varphi(\theta) = (\varphi_1(\theta),\cdots,\varphi_n(\theta)) \in \mathbb{E}^n$, we also have $\int_{-r}^{0} d\alpha(\theta)\varphi(\theta) = \sum_{j=1}^{n} \int_{-r}^{0} d\alpha_j(\theta)\varphi_j(\theta)$.
So, for $\varphi = \sum_{i=1}^{k} c_i\chi_{[\theta,0]}$ in $\mathcal{E}^+, c_i = (c_i^1, c_i^2, \cdots, c_i^n) \in \mathbb{E}^n$, we can take $\varphi_m = \sum_{i=1}^{k} c_i\chi^{(m)}_{[\theta_i,0]}$ in $\mathcal{C}$.
Then, for $\alpha \in G^-BV_0$ we have

$$\int_{-r}^{0} d\alpha(\beta)\varphi_m(\beta) = \sum_{j=1}^{n}\sum_{i=1}^{k} c_i^j \int_{-r}^{0} d\alpha_j(\beta)\chi^{(m)}_{[\theta_i,0]}(\beta)$$

$$\overset{m\to\infty}{\longrightarrow} \sum_{j=1}^{n}\sum_{i=1}^{k} c_i^j \int_{-r}^{0} d\alpha_j(\beta)\chi_{[\theta_i,0]}(\beta) = \int_{-r}^{0} d\alpha(\beta)\varphi(\beta) \qquad \text{(by item 1))}$$

We see that for the φ_m above we have $\varphi_m(0) = \varphi(0)$ and $\|\varphi_m\| = \|\varphi\|$ $\forall m \in \mathbb{N}$

Lemma 2. *For $\varphi \in \mathcal{G}^+$, if $\int_{-r}^{0} d\alpha(\beta)\varphi(\beta) = 0$ $\forall \alpha \in G^-BV_0$, then $\varphi(\theta) = 0$ for $-r \leq \theta < 0$.*

Proof. As in lemma 1, we can suppose, without loss of generality, that $n = 1$.

If we have for some $\theta \neq 0$ that $\varphi(\theta) \neq 0$, then $\exists\delta > 0$ such that $\varphi(\theta + t) \neq 0$ for $0 \leq t \leq \delta$. Take $\alpha = \chi_{[-r,\theta+\frac{\delta}{2}]}$ and we will have $\int_{-r}^{0} d\alpha(\beta)\varphi(\beta) = -\varphi(\theta + \frac{\delta^+}{2}) \neq 0$, which is a contradiction.

For a linear operator L, we denote by $\mathcal{N}(L)$ and $\mathcal{R}(L)$ the kernel and the range, respectively.

Remark 1. In [4], ch. 12.3, there is given a continuous projection $\Psi : \mathcal{C} \to \mathcal{C}_{D_0}$ such that $\Psi = I_{\mathcal{C}} - \Phi D^0$ where $\Psi = (\phi_1, \cdots, \phi_n), \phi_i \in \mathcal{C}$, satisfies $D^0\Phi = I$, I is the $n \times n$-identity matrix, and $I_{\mathcal{C}}$ is the identity of $\mathcal{C}$.

So, $\mathcal{C} = \mathcal{C}_{D^0} \oplus \mathcal{N}(\Psi)$ and $\dim\mathcal{N}(\Psi) = n$ because $\mathcal{N}(\Psi) = \mathcal{R}(\Phi D^0)$ has $\Phi = (\phi_1, \cdots, \phi_n)$ as a basis. Putting $\varphi^0 = \Psi\varphi$, we have, for $\varphi \in \mathcal{C}$:

$$\varphi = \varphi^0 + \Phi D^0\varphi = \varphi^0 + \sum_{i=1}^{n}(D^0\varphi)_i\phi_i = \varphi^0 + \sum_{i=1}^{n}(\varphi(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi(\theta))\phi_i$$

where $(D^0\varphi)_i$ is the i-th component of the vector $D^0\varphi \in \mathbb{E}^n$ and $\overline{\mu}_i(\theta)$ is the i-th line of the matrix $\overline{\mu}(\theta) = -\sum_{k=1}^{\infty} A_k\chi_{]-\infty,-r_k]}(\theta)$. Thus, $\overline{\mu}_i \in G^-BV_0, i = 1, 2, \cdots, n$.

For $\varphi \in \mathcal{G}^+$ it is also true that $D^0(\varphi - \Phi D^0\varphi) = D^0\varphi - D^0\Phi D^0\varphi = 0$. Therefore we can extend $\Psi : \mathcal{G}^+ \to \mathcal{G}^+_{D^0}$ as $\Psi = I_{\mathcal{G}^+} - \Phi D^0, I_{\mathcal{G}^+}$ being the identity of $\mathcal{G}^+$, and we will have $\mathcal{G}^+ = \mathcal{G}^+_{D^0} \oplus \mathcal{N}(\Psi)$, where the kernel $\mathcal{N}(\Psi)$ remains the same n-dimensional subspace of $\mathcal{C}$, that is, $\mathcal{N}(\Psi) = \mathcal{R}(\Phi D^0)$.

From this remark and lemma 1, it follows easily the:

Lemma 3. *For $\varphi \in \mathcal{E}^+$, let $\varphi_m \in \mathcal{C}, m \in \mathbb{N}$, as in lemma 1. Then*

$$\int_{-r}^{0} d\alpha(\beta)\varphi^0_m(\beta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\beta)\varphi^0(\beta) \qquad \forall\alpha \in G^-BV_0,$$

where $\varphi^0 = \Psi\varphi$ as in remark 1. We also have $\varphi^0_m(0) \overset{m\to\infty}{\longrightarrow} \varphi^0(0)$.

Proof. As in remark 1,

$$\varphi^0_m = \varphi_m - \sum_{i=1}^{n}(\varphi_m(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi_m(\theta))\phi_i.$$

Then, for $\alpha \in G^-BV_0$,

$$\int_{-r}^{0} d\alpha(\beta)\varphi^0_m(\beta) = \int_{-r}^{0} d\alpha(\beta)\varphi_m(\beta) - \sum_{i=1}^{n}\left(\varphi_m(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi_m(\theta)\right)\int_{-r}^{0} d\alpha(\beta)\phi_i(\beta).$$

Considering that $\varphi_m(0)_i = \varphi(0)_i$, and $\overline{\mu}_i \in G^-BV_0$ and using lemma 1, we have:

$$\int_{-r}^{0} d\alpha(\beta)\varphi^0_m(\beta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\beta)\varphi^0(\beta)$$

and also

$$\varphi_m^0(0) = \varphi_m(0) - \sum_{i=1}^{n}\left(\varphi_m(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi_m(\theta)\right)\phi_i(0)$$

$$=\varphi(0) - \sum_{i=1}^{n}\left(\varphi(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi_m(\theta)\right)\phi_i(0) \overset{m\to\infty}{\longrightarrow} \varphi^0(0)$$

Note that $\mathcal{E}_{D^0}^+ \overset{def.}{=} \Psi(\mathcal{E}^+)$ is dense in $\mathcal{G}_{D^0}^+$, since $\mathcal{E}^+$ is dense in $\mathcal{G}^+$ (see [8]) and Ψ is continuous from $\mathcal{G}^+$ onto $\mathcal{G}_{D^0}^+ = \Psi(\mathcal{G}^+)$.

Lemma 4. *If $T \in \mathcal{L}(\mathcal{G}_{D^0}^+, \mathcal{G}^+)$ with $(T\varphi)(\theta) = \int_{-\overline{r}}^{0} d_\beta K(\theta - \beta)\varphi(\beta)$, where we have $-r \leq -\overline{r} < 0$ and $K : [-r, \overline{r}] \to \mathcal{L}(\mathbb{E}^n)$ has bounded variation and is right-continuous; then, for each $\alpha \in G^-BV_0$, there is a $\widetilde{\alpha} \in G^-BV_0$, such that:*

$$\int_{-r}^{0} d\alpha(\theta)(T\varphi)(\theta) = \int_{-r}^{0} d\widetilde{\alpha}(\beta)\varphi(\beta) \qquad \forall\varphi \in \mathcal{G}_{D^0}^+.$$

Proof. We use the theorem 2.4 of [9], which says that:

$$\int_{-r}^{0} d\alpha(\theta)\int_{-\overline{r}}^{0} d_\beta K(\theta - \beta)\varphi(\beta) = \int_{-\overline{r}}^{0} d_\beta\left[\int_{-r}^{0} d\alpha(\theta)K(\theta - \beta)\right]\varphi(\beta)$$

to construct a suitable $\widetilde{\alpha} \in G^-BV_0$.

Remark 2. In [3] we show that the variation-of-constants formula for the linear NFDEs ([4], [6]) remains the same in the context of regulated functions. For $\varphi \in \mathcal{G}_{D^0}^+$, the solution y for $(D)_0$, for $t \geq 0$, with $y_0 = \varphi$ is given by

$$y(t) = -\sum_{k=1}^{\infty}\int_{-r_k}^{0} d_\beta X(t - \beta - r_k)A_k\varphi(\beta), \quad t \geq 0, \qquad (\rho_{D^0})$$

where X is the fundamental matrix given by the conditions $D^0X_t = I$ for $t \geq 0$, $X(0) = I$ and $X(t) \equiv 0$ for $t < 0$.

We have, by [6], lemma 3.5, the following result: if $\alpha \in \mathbb{R}$ is such that $\det H(\lambda) \neq 0$ in some strip $|Re\lambda - \alpha| \leq \delta$, $\delta > 0$, then we may decompose $X(t) = X^P(t) + X^Q(t)$ (if $\alpha = 0$ we will have $X(t) = X^P(t) + X^Q(t)+$ constant), X^P can be extended for $t \leq 0$ and we have the estimates:

$$Var_{[t-r,t]}[X^Q] \leq Me^{(\alpha-\delta)t} \qquad \text{for } t \geq 0$$
$$Var_{[t-r,t]}[X^P] \leq Me^{(\alpha+\delta)t} \qquad \text{for } t \geq 0$$

for some constant M.

Theorem 1. *Suppose $\alpha \in \mathbb{R}$ is such that det $H(\lambda) \neq 0$ in some strip $|Re\lambda - \alpha| \leq \delta, \delta > 0$. Then there exist closed subspaces P, Q of $\mathcal{G}^+_{D^0}$, invariant under $T^0(t), t \geq 0$, such that $\mathcal{G}^+_{D^0} = P \oplus Q$, where $\oplus$ means the direct sum. The semigroup $\{T^0(t)_{|P}\}_{t\geq 0}$ may be extended uniquely as a group $(-\infty < t < \infty)$ of operators on P. There exists a constant M' such that : $\|T^0(t)_{|Q}\| \leq M'.e^{(\alpha-\delta)t}$ for $t \geq 0$ and $\|T^0(t)_{|P}\| \leq M'.e^{(\alpha+\delta)t}$ for $t \leq 0$.*

Proof. As in remark 2, we have the split of matrix X and, for $\varphi \in \mathcal{G}^+_{D^0}$, we have the solution of $(D)_0,, y(t)$, given by formula (ρ_{D^0}). We may write: $y(t) = y^P(t) + y^Q(t)$, where

$$y^{P,Q}(t) \overset{def.}{=} -\sum_{k=1}^{\infty} \int_{-r_k}^{0} d_\beta X^{P,Q}(t-\beta-r_k) A_k \varphi(\beta), \qquad t \geq 0,$$

and we can take $-\infty < t < \infty$ for y^P. Since $X^P : \mathbb{R} \to \mathcal{L}(\mathbb{E}^n)$ and $X^Q : \mathbb{R}_+ \to \mathcal{L}(\mathbb{E}^n)$ are right-continuous and of bounded variation in each compact interval of t (see [6] lemma 3.5), we have that $y^{P,Q}$ are well defined and are right-continuous regulated functions because: I) each Interior Integral $\int_{-r_k}^{0} d_\beta \cdots$ in the formula defines a right-continuous regulated function of t (see [9] §2), so $y^{P,Q}_N(t) \overset{def.}{=} -\sum_{k=1}^{N} \int_{-r_k}^{0} d_\beta X^{P,Q}(t-\beta-r_k) A_k \varphi(\beta)$ is also right-continuous regulated in t, for each $N \in \mathbb{N}$, and II) $\{y^P_N\}_{N\in\mathbb{N}}$ and $\{y^Q_N\}_{N\in\mathbb{N}}$ are Cauchy-sequences in the space of right-continuous regulated functions when we take t in any compact interval, with the uniform-norm, since $\sum_{k=1}^{\infty} |A_k| < \infty$, and then they converge as these spaces are complete (see [8]).

From remark 2, we also obtain the estimates:

$$\begin{cases} \|y^Q(t)\| \leq (\sum_{k=1}^{\infty} |A_k|).M.e^{(\alpha-\delta)t}\|\varphi\| & \text{for } t \geq 0 \\ \|y^P(t)\| \leq (\sum_{k=1}^{\infty} |A_k|).M.e^{(\alpha+\delta)t}\|\varphi\| & \text{for } t \leq 0 \end{cases} \qquad (*)$$

Define $T^0(t)^P \varphi = y^P_t$ for $t \in \mathbb{R}$ and $T^0(t)^Q \varphi = y^Q_t$ for $t \geq r$. By the majorations above we see that $T^0(t)^{P,Q} \in \mathcal{L}(\mathcal{G}^+_{D^0}, \mathcal{G}^+)$. Let $\pi_P \overset{def.}{=} T^0(0)^P \in \mathcal{L}(\mathcal{G}^+_{D^0}, \mathcal{G}^+)$.

In [6] theorem 3.1, it is shown that $\pi_P|_{\mathcal{C}_{D^0}} \in \mathcal{L}(\mathcal{C}_{D^0})$ and it is idempotent.

We will show that $\pi_P \in \mathcal{L}(\mathcal{G}^+_{D^0})$ and is also idempotent.

We begin with the step-functions. Let $\varphi \in \mathcal{E}^+$ and $\varphi_m \in \mathcal{C}, m \in \mathbb{N}$, as in lemma 1. By remark 1, we have: $\varphi = \varphi^0 + \sum_{i=1}^{n}(\varphi(0)_i - \int_{-r}^{0} d\overline{\mu}_i(\theta)\varphi(\theta))\phi_i$. Using the formula of $\pi_P\varphi^0 = y^P_0$ (now y is the solution of $(D)_0$ with initial value φ^0) and lemmas 3 and 4 we obtain $\int_{-r}^{0} d\alpha(\beta)\pi_P\varphi^0_m(\beta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^{0} d\alpha(\beta)\pi_P\varphi^0(\beta)$ $\forall \alpha \in G^-BV_0$. By the formula of $\pi_P\varphi^0_m(\theta)$, for each $\theta \in [-r, 0]$, and the fact that $\|\varphi_m\| = \|\varphi\|$ (and then $\|\varphi^0_m\| \leq \|\Psi\|.\|\varphi\|$ $\forall_m \in \mathbb{N}$), we obtain $\pi_P\varphi^0_m(\theta) \overset{m\to\infty}{\longrightarrow} \pi_P\varphi^0(\theta)$ and, in particular, $\pi_P\varphi^0_m(0) \overset{m\to\infty}{\longrightarrow} \pi_P\varphi^0(0)$. Since $\varphi^0_m = \Psi(\varphi_m) \in \mathcal{C}_{D^0}$, we have, by [6] theorem 3.1, that $\pi_P\varphi^0_m \in \mathcal{C}_{D^0}$, so $0 = D^0(\pi_P\varphi^0_m) = \pi_P\varphi^0_m(0) - \int_{-r}^{0} d\overline{\mu}(\beta)(\pi_P\varphi^0_m)(\beta) \overset{m\to\infty}{\longrightarrow} \pi_P\varphi^0(0) - \int_{-r}^{0} d\overline{\mu}(\beta)(\pi_P\varphi^0)(\beta) = D^0(\pi_P\varphi^0)$. Then $\pi_P\varphi^0 \in \mathcal{G}^+_{D^0}$, that is, $\pi_P(\mathcal{E}^+_{D^0}) \subset \mathcal{G}^+_{D^0}$ and taking the closure of $\mathcal{E}^+_{D^0}$, we have $\pi_P \in \mathcal{L}(\mathcal{G}^+_{D^0})$. Now it makes sense to take π_P^2. To show that π_P is a projection, we note first that $\int_{-r}^0 d\alpha(\beta)(\pi_P^2\varphi_m^0)(\beta) \overset{m\to\infty}{\longrightarrow} \int_{-r}^0 d\alpha(\beta)(\pi_P^2\varphi^0)(\beta)$, $\forall\alpha \in G^-BV_0$, a consequence of the formula of $\pi_P^2\varphi^0(\beta)$ and lemma 4. Note that

$$\pi_P^2\varphi^0(\beta) = (T^0(0)^P[T^0(0)^P\varphi^0])(\beta) = -\sum_{k=1}^{\infty}\int_{-r_k}^{0} d_\tau X^P(\beta-\tau-r_k)A_k[y^P(\tau)] =$$

$$-\sum_{k=1}^{\infty}\int_{-r_k}^{0} d\tau X^P(\beta-\tau-r_k)A_k[-\sum_{j=1}^{\infty}\int_{-r_j}^{0} d_\sigma X^P(\tau-\sigma-r_j)A_j\varphi^0(\sigma)].$$

We know that $\pi_P^2\varphi_m^0 = \pi_P\varphi_m^0$, since $\varphi_m^0 \in \mathcal{C}_{D^0}$. Therefore, $\int_{-r}^0 d\alpha(\beta)[\pi_P^2 - \pi_P]\varphi^0(\beta) = 0$ $\forall\alpha \in G^-BV_0$ and from lemma 2 we have $\pi_P^2\varphi^0(\theta) = \pi_P\varphi^0(\theta)$ for $-r \le \theta < 0$. For $\theta = 0$, observe that $D^0(\pi_P^2\varphi^0) = \pi_P^2\varphi^0(0) - \int_{-r}^0 d\overline{\mu}(\theta)\pi_P^2\varphi^0(\theta) = \pi_P^2\varphi^0(0) - \pi_P\varphi^0(0) + D^0(\pi_P\varphi^0)$, but $D^0(\pi_P^2\varphi^0) = D^0(\pi_P\varphi^0) = 0$ since $\pi_P \in \mathcal{L}(\mathcal{G}^+_{D^0})$, that is, $\pi_P(\mathcal{G}^+_{D^0}) \subset \mathcal{G}^+_{D^0} = \mathcal{N}(D^0)$. This completes the proof that π_P is idempotent in $\mathcal{E}^+_{D^0}$ and so in $\mathcal{G}^+_{D^0}$.

Then we have the closed subspaces of $\mathcal{G}^+_{D^0} : P = \mathcal{R}(\pi_P), Q = \mathcal{N}(\pi_P)$; $\mathcal{G}^+_{D^0} = P \oplus Q$ and π_P is a projection on P along Q.

By [6] theorem 3.1, we have $T^0(t)\pi_P\varphi_m^0(\theta) = \pi_P T^0(t)\varphi_m^0(\theta) = T^0(t)^P\varphi_m^0(\theta)$ $\forall t \ge 0$, $\forall\theta \in [-r, 0]$ and each of these expressions converges when $m \to \infty$ to the respective expression with φ^0 instead of φ_m^0 (this can be shown by using the formulas of $\pi_P, T^0(t), T^0(t)^P$ and lemma 4). Therefore, $T^0(t)\pi_P = \pi_P T^0(t) = T^0(t)^P$ in $\mathcal{E}^+_{D^0}$ and in $\mathcal{G}^+_{D^0}$, for $t \ge 0$. For $t \in \mathbb{R}$, we also obtain $0 = D^0(T^0(t)^P\varphi_m^0) \overset{m\to\infty}{\longrightarrow} D^0(T^0(t)^P\varphi^0)$ and this allows us to define $T^0(t) = T^0(t)^P$ in P for $t \le 0$ and to obtain the group of isomorphisms $\{T^0(t)_{|P}\}_{t\in\mathbb{R}}$; for, when we have the backward continuation of the solution of equations like $(D)_0$ in the whole line, this continuaiton is unique (see [5]).

The inequalities stated in the theorem follow immediately from inequalities in $(*)$.

Remark 3. The subspaces $P \cap \mathcal{C}_{D^0}$ and $Q \cap \mathcal{C}_{D^0}$ are characterized in [6] theorem 3.1, in terms of generalized eigenspaces corresponding to the eigenvalues of the infinitesimal generator $\mathbf{A}^0$ which have the real parts bigger than α and smaller than α, respectively. We can extend, now , the theorem 3.2 of [6],

Theorem 2. *Assume that $\lambda \mapsto detH(\lambda)$ has zeros; then, for $t \ge 0$, we have:*

$$\overline{\{e^{\lambda t} | detH(\lambda) = 0\}} \subset \sigma(T^0(t)) \subset \{\mu \mid |\mu| = e^{\xi t}, \xi \in \overline{\mathcal{Z}}\} \cup \{0\},$$

where $\mathcal{Z} \overset{def.}{=} \{Re\lambda \mid det\ H(\lambda) = 0\}$ and $\overline{\mathcal{Z}}$ is the closure of $\mathcal{Z}$.

If $\alpha \notin \overline{\mathcal{Z}}$ and $\mathcal{G}^+_{D^0} = P \oplus Q$ is the decomposition given by theorem 1, then

$$\sigma(T^0(t)_{|P}) \subset \{\mu \mid |\mu| = e^{\xi t}, \xi \in \overline{\mathcal{Z}} \text{ and } \xi > \alpha\}$$
$$\sigma(T^0(t)_{|Q}) \subset \{\mu \mid |\mu| = e^{\xi t}, \xi \in \overline{\mathcal{Z}} \text{ and } \xi < \alpha\}$$

both for $t \ge 0$.

If $\mathcal{Z}$ is empty, then $\sigma(T^0(t)) = \{0\}$ for $t > 0$; in fact, $T^0(t) = 0$ for $t \geq r.n.$

Proof. It is the same as for theorem 3.2 of [6], using now theorem 1. We recall that for $\varphi \in \mathcal{E}^+$ and $\varphi_m \in \mathcal{C}, m \in \mathbb{N}$, as in lemma 1, we will have $T^0(t)\varphi_m^0(\theta) \overset{m\to\infty}{\longrightarrow} T^0(t)\varphi^0(\theta)$ and then $T^0(t)_{|\mathcal{C}_{D^0}} = 0 \Rightarrow T^0(t) = 0$.

Remark 4. We have from [7] theorem 5.1 that if V is any subset of the complex plane and $U = \{x \in \mathbb{R} \mid \exists$ sequence $\{z_k\}_{k\in\mathbb{N}}$, $z_k \in V$ with $Re\, z_k \to x$ and $|Im\, z_k| \to \infty$ as $k \to \infty\}$, then for almost all real t, the inclusion $e^{t(U+i\mathbb{R})} \subset \overline{e^{tV}}$ holds, where $U + i\mathbb{R} = \{z \in \mathbb{C} | Re\, z \in U\}$. To see that this inclusion may not hold for all t, let us consider $V = \{0, \pm i, \pm 2i, \pm 3i, \cdots\}$ so $U = \{0\}, e^{t(i\mathbb{R})}$ is the unit circle for $t \neq 0$ and we note that $e^{t(i\mathbb{R})} \subset \overline{e^{tV}}$ if and only if t/π is irrational.

Since det $H(\lambda)$ is an analytic almost periodic function of λ, we have from [6] lemma 3.2 that if det $H(\lambda) = 0$ for some $\lambda \in \mathbb{C}$, then there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that det $H(\lambda_k) = 0, |\lambda_k| \to \infty$ and $Re\lambda_k \to Re\lambda$ as $k \to \infty$. If we take $V = \sigma(\mathbf{A}^0) = \{\lambda \in \mathbb{C} \mid \det H(\lambda) = 0\}$, then we have $U = \overline{\mathcal{Z}}$ and $e^{t(\overline{\mathcal{Z}}+i\mathbb{R})} = \{\mu \mid |\mu| = e^{\xi t}, \xi \in \overline{\mathcal{Z}}\}$ and from theorem 2 and the above result from [7], we conclude that

$$\sigma(T^0(t))\backslash\{0\} = \overline{e^{t\sigma(\mathbf{A}^0)}}\backslash\{0\}$$

for almost all $t \geq 0$.

4 The Neutral FDE

Passing now to equation (N) of section 2, we first generalize the lemma 4.1 of [6].

Lemma 1. *For the equations (N) and $(D)_0$ and their flows, given in section 2, we have: $T(t) - T^0(t) \circ \Psi : \mathcal{G}^+ \to \mathcal{G}^+$ is a compact operator for each $t \geq 0$, where Ψ is the projection given in remark 1 of section 3.*

Proof. Analogous to the lemma 4.1 of [6]. Recall that $\mathcal{R}(I_{\mathcal{G}^+} - \Psi)$ has finite dimension.

We denote by $P\sigma(L), R\sigma(L)$ and $C\sigma(L)$ the point, the residual and the continuous parts of the spectrum of a linear operator L.

We generalize now the theorem 4.1 of [6].

Theorem 1. *With the notation of section 2, for the flow of equation (N), we have:*

i) $P\sigma(T(t))\backslash\{0\} = P\sigma(T(t)_{|\mathcal{C}})\backslash\{0\} = \{e^{\lambda t} | det\Delta(\lambda) = 0\}$

ii) $R\sigma(T(t)) \cup C\sigma(T(t)) \subset \{\mu \mid |\mu| = e^{\xi t}, \xi \in \overline{\mathcal{Z}}\} \cup \{0\}$ *where $\mathcal{Z}$ is given in theorem 2 of section 3.*

iii) $\sigma(T(t))\backslash\{0\} = \overline{e^{t\sigma(A)}}\backslash\{0\}$ *a.e. in $t \geq 0$.*

Proof. i) Suppose $t_0 > 0, \lambda_0 \in \mathbb{C}$ and $\varphi \in \mathcal{G}^+$ such that $T(t_0)\varphi = e^{\lambda_0 t_0}\varphi \neq 0$.

We show that there is a $c \in \mathbb{E}^n$ such that $z(t) = c.e^{\lambda t} \neq 0$ is the solution of (N) with initial data $\psi(\theta) = c.e^{\lambda\theta}$, where $\lambda = \lambda_0 + \frac{2\pi i m}{t_0}$ for some $m \in \mathbb{N}$, that is; we find a continuous (in fact, exponential) eigenvector for the eigenvalue $e^{\lambda_0 t_0}$ of $T(t_0)$.

Let $x_t = T(t)\varphi, t \geq 0$. The function $t \mapsto e^{-\lambda_0 t}x(t)$ is periodic of period t_0 and then there is a $m \in \mathbb{N}$ such that the m-th Fourier coefficient is nonzero, that is,

$$c = \frac{1}{t_0}\int_0^{t_0} e^{\frac{-2\pi i m}{t_0}s}x(s)e^{-\lambda_0 s}ds = \frac{1}{t_0}\int_0^{t_0} x(s)e^{-\lambda s}ds \neq 0$$

and we have

$$\begin{aligned}
ce^{\lambda(t+\theta)} =& \frac{1}{t_0}\int_0^{t_0} x(s)e^{\lambda(t+\theta-s)}ds \\
=& \frac{1}{t_0}\int_0^{t_0} x(t+u+\theta)e^{-\lambda u}du \\
=& \frac{1}{t_0}\int_0^{t_0} [T(t)x_u](\theta)e^{-\lambda u}du \\
=& [T(t)(\frac{1}{t_0}\int_0^{t_0} x_u(\cdot)e^{-\lambda u}du)](\theta) = [T(t)(c.e^{\lambda\cdot})](\theta).
\end{aligned}$$

To prove the last equalities, let Δ be any partition of interval $[0, t_0]$ $(0 = u_0 < u_1 \cdots < u_k = t_0, \overline{u}_i \in [u_{i-1}, u_i], \Delta u_i = u_i - u_{i-1})$ and we have:

$$[T(t)(\frac{1}{t_0}\sum_{i=1}^{k} x_{\overline{u}_i}(\cdot)e^{-\lambda\overline{u}_i}\Delta u_i)](\theta) = \frac{1}{t_0}\sum_{i=1}^{k}[T(t)x_{\overline{u}_i}](\theta)e^{-\lambda\overline{u}_i}\Delta u_i \to$$

$$\xrightarrow{\|\Delta\|\to 0} \frac{1}{t_0}\int_0^{t_0} [T(t)x_u](\theta)e^{-\lambda u}du, \text{ but}$$

$$\| [T(t)(\frac{1}{t_0}\int_0^{t_0} x_u(\cdot)e^{-\lambda u}du)](\theta) - [T(t)(\frac{1}{t_0}\sum_{i=1}^{k} x_{\overline{u_i}}(\cdot)e^{-\lambda\overline{u_i}}\Delta u_i)](\theta)\|$$

$$\leq \|T(t)\| \cdot \frac{1}{t_0} \cdot \| \int_0^{t_0} x_u(\cdot)e^{-\lambda u}du - \sum_{i=1}^{k} x_{\overline{u_i}}(\cdot)e^{-\lambda\overline{u}_i}\Delta u_i\|$$

$$
\begin{aligned}
&=\|T(t)\|\frac{1}{t_0}\sup_{\overline{\theta}\in[-r,0]}\|\int_0^{t_0}x_u(\overline{\theta})e^{-\lambda u}du-\sum_{i=1}^{k}x_{\overline{u_i}}(\overline{\theta})e^{-\lambda\overline{u}_i}\Delta u_i\|\\
&\leq\underbrace{\|T(t)\|\frac{1}{t_0}\sup_{\overline{\theta}\in[-r,0]}\|e^{\lambda\overline{\theta}}\|}_{K}\cdot\sup_{\overline{\theta}\in[-r,0]}\|\int_0^{t_0}x(u+\overline{\theta})e^{-\lambda(u+\overline{\theta})}du\\
&\quad-\sum_{i=1}^{k}x(\overline{u}_i+\overline{\theta})e^{-\lambda(\overline{u}_i+\overline{\theta})}\Delta u_i\|\\
&\leq K\cdot\sup_{\overline{\theta}\in[-r,0]}\|\int_{\overline{\theta}}^{t_0+\overline{\theta}}x(s)e^{-\lambda s}ds-\sum_{i=1}^{k}x(\overline{u}_i+\overline{\theta})e^{-\lambda(\overline{u_i}+\overline{\theta})}\Delta u_i\|=K\cdot M(\Delta)
\end{aligned}
$$

Since the function $t\mapsto y(t)\overset{def.}{=}x(t)e^{-\lambda t}$ is periodic of period t_0, we have

$$
M(\Delta)=\sup_{\overline{\theta}\in[-r,0]\cap]-t_0,0]}\|\int_0^{t_0}y(s)ds-\sum_{i=1}^{k}y(\overline{u}_i+\overline{\theta})\Delta u_i\|.
$$

For each $\overline{\theta}\in[-r,0]\cap]-t_0,0]$, let us call $\Delta+\overline{\theta}$ the translation of Δ to the interval $[\overline{\theta},t_0+\overline{\theta}](\overline{\theta}=u_0+\overline{\theta}<u_1+\overline{\theta}\cdots<u_k+\overline{\theta}=t_0+\overline{\theta})$. Define $l=l(\overline{\theta})\in\{1,2,\cdots,k\}$ such that $u_{l-1}+\overline{\theta}\leq 0<u_l+\overline{\theta}$. We have the following situation (where we delete the first summation on the right hand side if $l=1$ and the last summation if $l=k$):

$$
\begin{aligned}
&\sum_{i=1}^{k}y(\overline{u}_i+\overline{\theta})\Delta u_i\\
&=\sum_{i=1}^{l-1}y(\overline{u}_i+\overline{\theta}+t_0)\Delta u_i+y(\overline{u}_l+\overline{\theta})\Delta u_l+\sum_{i=l+1}^{k}y(\overline{u}_i+\overline{\theta})\Delta u_i\\
&=\sum_{j=k-l+2}^{k}y(\overline{s}_j(\overline{\theta}))\Delta s_j(\overline{\theta})+y(t_0)\underbrace{[t_0-(u_{l-1}+\overline{\theta}+t_0)]}_{\Delta s_{k+1}(\overline{\theta})}+y(0)\underbrace{[u_l+\overline{\theta}-0]}_{\Delta s_1(\overline{\theta})}\\
&\quad+[y(\overline{u}_l+\overline{\theta})-y(0)][u_l-u_{l-1}]+\sum_{j=2}^{k-l+1}y(\overline{s}_j(\overline{\theta}))\Delta s_j(\overline{\theta})\\
&=\sum_{j=1}^{k+1}y(\overline{s}_j(\overline{\theta}))\Delta s_j(\overline{\theta})+[y(\overline{u}_l+\overline{\theta})-y(0)]\Delta u_l.
\end{aligned}
$$

This defines the partition $\Delta_{\overline{\theta}}$ of interval $[0,t_0]$ $(0=s_0(\overline{\theta})<s_1(\overline{\theta})\cdots<s_{k+1}(\overline{\theta})=t_0,\overline{s}_j(\overline{\theta})\in[s_{j-1}(\overline{\theta}),s_j(\overline{\theta})],\Delta s_j(\overline{\theta})=s_j(\overline{\theta})-s_{j-1}(\overline{\theta}))$ as shown in the figure

Therefore

$$M(\Delta) \leq \sup_{\overline{\theta} \in [-r,0] \cap]-t_0,0]} \| \int_0^{t_0} y(s)ds - \sum_{j=1}^{k+1} y(\overline{s}_j(\overline{\theta}))\Delta s_j(\overline{\theta})\| + 2 \sup_{s\in[0,t_0]} \|y(s)\| \cdot \|\Delta\|.$$

Since $\|\Delta_{\overline{\theta}}\| \leq \|\Delta\|$ for any $\overline{\theta}$ in $[-r,0]$, we see that $M(\Delta)$ goes to zero when $\|\Delta\| \to 0$. We have also $\frac{1}{t_0}\int_0^{t_0} x_u(\theta)e^{-\lambda u}du = \frac{1}{t_0}\int_0^{t_0} x(u+\theta)e^{-\lambda(u+\theta)}du.e^{\lambda\theta} = c.e^{\lambda\theta}$.

By the same arguments as for theorem 4.1 of [6] we prove ii), using the result of Gohberg and Krein in the version of lemma 2 of [6] and using theorem 2 of section 3 and lemma 1.

For iii) we use theorem 5.1 of [7].

In the same way as theorem 4.2 of [6] we can show that:

Theorem 2. *Suppose $\alpha \notin \overline{Re\sigma(\mathbf{A})}$, i.e., $\det \Delta(\lambda) \neq 0$ in some strip $|Re\lambda - \alpha| < \delta, \delta > 0$. Then $\mathcal{G}^+ = P \oplus Q$, where P, Q are closed subspaces invariant under $T(t)$.*

The restriction of the semigroup to P may be extended to a group $\{T(t)_{|P}\}_{t\in\mathbb{R}}$ of isomorphisms of P. Finally, there exists a constant M such that $\|T(t)_{|Q}\| \leq Me^{(\alpha-\delta)t}$ for $t \geq 0$ and $\|T(t)_{|P}\| \leq Me^{(\alpha+\delta)t}$ for $t \leq 0$ (see also [3], chap.II,§4, theorems 3 and 4.)

We now study the corresponding decomposition for the nonhomogeneous equation

$$\frac{d}{dt}(Dx_t - H(t)) = Lx_t \qquad (N)_H$$

for a given regulated right-continuous forcing function H.

For each $t, t_0 \in \mathbb{R}, t \geq t_0$, we have a bounded linear operator $\mathcal{K}(t,t_0) \in \mathcal{L}(G^+([t_0,t],\mathbb{E}^n), \mathcal{G}^+)$ such that the solution of $(N)_H$ for $t \geq t_0$ with $x_{t_0} = \varphi \in \mathcal{G}^+$ is given by $x_t(t_0,\varphi,H) = T(t-t_0)\varphi + \mathcal{K}(t,t_0)H$, where $\{T(t)\}_{t\geq 0}$ is the flow of (N) as in section 2.

From [3] we have that $\mathcal{K}(t,t_0)H = \chi_0 H(t) - T(t-t_0)\chi_0 H(t_0) - \int_{t_0}^t d_\sigma T(t-\sigma)\chi_0 H(\sigma)$ where, for $p \in \mathbb{E}^n$, we have $\chi_0 p(\theta) = 0$ for $-r \leq \theta < 0$ and $\chi_0 p(0) = p$ and $(\int_{t_0}^t d_\sigma T(t-\sigma)\chi_0 H(\sigma))(\theta) = \int_{t_0}^t d_\sigma X(t+\theta-\sigma)H(\sigma)$, the integral being in $\mathbb{E}^n$, and the fundamental matrix $X(t) \in \mathcal{L}(\mathbb{E}^n)$ given by $X(t)p = T(t)\chi_0 p(0)$.

In [3] (see chap.II, §4, remark 7), we generalize the theorem 4.3 of [6] as a consequence of theorem 2 and the variation-of-constants formula for equation $(N)_H$. Then we have:

Theorem 3 *Suppose $\alpha \notin \overline{Re\sigma(\mathbf{A})}$ and $\mathcal{G}^+ = P \oplus Q$ is the decomposition provided by theorem 2. We write $\varphi = \varphi^P + \varphi^Q \in P \oplus Q$. Then, there exist matrix-functions of bounded variation X^P and X^Q, such that:*

$$\begin{aligned} x_t^Q(t_0,\varphi,H) =& T(t-t_0)\varphi^Q + (\chi_0 H(t))^Q - T(t-t_0)(\chi_0 H(t_0))^Q \\ &- \int_{t_0}^t d_\sigma T(t-\sigma)[\chi_0 H(\sigma)]^Q \end{aligned}$$

and

$$x_t^P(t_0,\varphi,H) = T(t-t_0)\varphi^P + (\chi_0 H(t))^P - T(t-t_0)(\chi_0 H(t_0))^P - \int_{t_0}^t d_\sigma T(t-\sigma)[\chi_0 H(\sigma)]^P$$

for $t \geq t_0$, *where*

$$\left(\int_{t_0}^t d_\sigma T(t-\sigma)[\chi_0 H(\sigma)]^{P,Q}\right)(\theta) = \int_{t_0}^t d_\sigma X^{P,Q}(t+\theta-\sigma)H(\sigma)$$

for $\theta \in [-r,0]$.

Remark 2. The matrix functions X^Q and X^P are the same as in theorem 4.3 of [6] (our $X^{P,Q}(t+\theta)$ are his $T(t)\chi_0^{P,Q}(\theta)$) and we obtain the estimates given in that theorem.

References

[1] O. Diekmann, *Perturbed Dual Semigroups and Delay Equations*, Dynamics of Infinite Dimensional Systems (Lisbon 1986), 67-73. NATO Adv. Sci. Inst. Série F vol. 37, edited by Chow, S. N. and Hale, J. K.-Springer Berlin-New York, 1987.

[2] L. Fichmann, *Equações integrais de Volterra-Stieltjes e Equações do tipo neutro*, Tese de mestrado, IME-USP, Brasil, 1984.

[3] L.Fichmann, *Equações Diferenciais Neutras com condições iniciais descontinuas*, Tese de doutorado, IME-USP, Brasil, 1991.

[4] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, 1977.

[5] D. Henry, "Small solutions of linear autonomous FDEs," *J. Differential Equations* 8 (1970).

[6] D. Henry, "Linear Autonomous Neutral Functional Differential Equations," *Journal of Differential Equations* 15 (106-128), 1974.

[7] D. Henry, "A property of the exponential function," *Publ .Mat*, UAB (Universitat Autônoma de Barcelona), vol.31, nº 1, artigo 5-Bellaterra, abril 1987.

[8] C.S. Hönig, *Volterra-Stieltjes Integral Equations: Functional Analytic Methods, Linear Constraints*, Amsterdam, North-Holland, 1975 (North-Holland Math. Studies, 16).

[9] C.S. Hönig, *Equations Integrales generalisées et aplications*, Probl. Mathématiques d'Orsay, Exposé nº 5, 1981-82.

[10] J.C.F. Oliveira and L. Fichmann, "Discontinuous solutions of Neutral Functional Differential Equations," *Publ. Mat. UAB* (Universitat Autonoma de Barcelona) Vol. 37, no.2, (369-386), 1993, Bellaterra.

[11] S. Schwabick, "A survey of some new results for regulated functions," *28º Seminário Brasileiro de Análise*, 1988, SBM, p.201-209.

Departamento de Matemática - IME-Universidade de São Paulo Caixa Postal 20570 - 01452-990 - São Paulo - SP - Brasil.