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ON A CERTAIN CONVOLUTION OPERATOR FOR MEROMORPHIC FUNCTIONS

MING-PO CHEN AND SHIGEYOSHI OWA

Abstract. A certain convolution operator $D^\alpha f(z)$ for meromorphic functions and two subclasses $M(\alpha)$ and $J(\alpha)$ are introduced. The object of the present paper is to derive some properties of the classes $M(\alpha)$ and $J(\alpha)$. The results of the paper are the generalizations and the improvements of the former results.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

which are regular in the annulus $E = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue one there. For functions

$$f_j(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,j} z^n \qquad (j = 1, 2) \tag{1.2}$$

in the class Σ, we define the convolution of $f_1(z)$ and $f_2(z)$ by

$$f_1 * f_2(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.3}$$

Using the convolution, we introduce the following convolution operator $D^\alpha f(z)$ by

$$D^\alpha f(z) = \frac{1}{z(1-z)^{\alpha+1}} * f(z) \qquad (\alpha > -1) \tag{1.4}$$

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for $f(z) \in \Sigma$. A function $f(z) \in \Sigma$ is said to be in the class $M(\alpha)$ if it satisfies

$$Re\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} < \frac{\alpha+2}{\alpha+1} \tag{1.5}$$

for some $\alpha(\alpha > -1)$ and for all $z \in U = \{z : |z| < 1\}$. The class $M(\alpha)$ when $\alpha = n \in N_0 = \{0, 1, 2, \ldots\}$ was introduced by Ganigi and Uralegaddi [1].

Theorem A ([1]). *$M(n+1) \subset M(n)$ for each $n \in N_0$.*

Further, a function $f(z) \in \Sigma$ is said to be a member of the class $J(\alpha)$ if it satisfies

$$Re\left\{\frac{(D^{\alpha+1}f(z))'}{(D^{\alpha}f(z))'}\right\} < \frac{\alpha+2}{\alpha+1} \tag{1.6}$$

for some $\alpha(\alpha > -1)$ and for all $z \in U$. The class $J(\alpha)$ when $\alpha = n \in N_0$ was also studied by Uralegaddi and Ganigi [4].

Theorem B ([4]). *$J(n+1) \subset J(n)$ for each $n \in N_0$.*

In the present paper, we give the improvements of the above theorems.

2. The Class $M(\alpha)$

We begin with the statement of the following lemmas.

Lemma 1 ([3]). *Let $\varphi(u,v)$ be a complex valued function,*

$$\varphi : D \to C, D \subset C^2 \qquad (C \text{ is the complex plane }),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u,v)$ satisfies

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\{\varphi(1,0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$, $Re\{\varphi(iu_2, v_1)\} \leq 0$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\varphi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then $Re\{p(z)\} > 0 \quad (z \in U)$.

Lemma 2 ([2]). *Let $w(z)$ be regular in U with $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, we have*

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geq 1$.

Now, we derive

Theorem 1. *If $f(z) \in M(\alpha+1)$, $\alpha > -1$ and $1 < \beta(\alpha) < \frac{\alpha+2}{\alpha+1}$, then*

$$Re\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} < \beta(\alpha) \qquad (z \in U), \tag{2.1}$$

where

$$\beta(\alpha) = \frac{2\alpha+3+\sqrt{(2\alpha+3)^2+8(\alpha+1)}}{4(\alpha+1)}. \tag{2.2}$$

Proof. For $f(z) \in M(\alpha+1)$, we define the function $p(z)$ by

$$\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} = \beta + (1-\beta)p(z) \tag{2.3}$$

with $\beta = \beta(\alpha)$. Then $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in U. With the help of the identity

$$z(D^{\alpha}f(z))' = (\alpha+1)D^{\alpha+1}f(z) - (\alpha+2)D^{\alpha}f(z), \tag{2.4}$$

(2.3) gives that

$$\begin{aligned}&\frac{\alpha+3}{\alpha+2} - \frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}\\ =&\frac{1}{\alpha+2}\{(\alpha+2)-(\alpha+1)(\beta+(1-\beta)p(z)) - \frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\},\end{aligned} \tag{2.5}$$

that is, that

$$\begin{aligned}&Re\left\{\frac{\alpha+3}{\alpha+2} - \frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}\right\}\\ =&Re\frac{1}{\alpha+2}\left\{(\alpha+2)-(\alpha+1)(\beta+(1-\beta)p(z)) - \frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\right\} > 0.\end{aligned} \tag{2.6}$$

Now, we define the function $\varphi(u,v)$ by

$$\varphi(u,v) = \frac{1}{\alpha+2}\left\{(\alpha+2)-(\alpha+1)(\beta+(1-\beta)u) - \frac{(1-\beta)v}{\beta+(1-\beta)u}\right\}, \tag{2.7}$$

which satisfies $Re\{\varphi(p(z), zp'(z))\} > 0$ for $z \in U$. Also we see that

(i) $\varphi(u,v)$ is continous in $D = (C - \{\frac{\beta}{\beta-1}\}) \times C$;

(ii) $(1,0) \in D$ and $Re\{\varphi(1,0)\} = \frac{1}{\alpha+2} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1+u_2^2}{2}$,

$$\begin{aligned}Re\{\varphi(iu_2,v_1)\} =&\frac{1}{\alpha+2}\left\{(\alpha+2)-(\alpha+1)\beta - \frac{\beta(1-\beta)v_1}{\beta^2+(1-\beta)^2U_2^2}\right\}\\ \leq&\frac{1}{\alpha+2}\left\{(\alpha+2)-(\alpha+1)\beta + \frac{\beta(1-\beta)(1+u_2^2)}{2(\beta^2+(1-\beta)^2u_2^2)}\right\}\\ =&\frac{(1-\beta)\{3+2(\alpha+1)(1-\beta)\}}{2(\alpha+2)(\beta^2+(1-\beta)^2u_2^2)}u_2^2 \leq 0,\end{aligned}$$

because $2(\alpha+1)\beta^2-(2\alpha+3)\beta-1=0$, $\alpha>-1$ and $1<\beta<\frac{\alpha+2}{\alpha+1}<\frac{2\alpha+5}{2(\alpha+1)}=1+\frac{3}{2(\alpha+1)}$. Thus the function $\varphi(u,v)$ satisfies the conditions in Lemma 1. Therefore we have that $Re\{p(z)\}>0$ $(z\in U)$, or, that

$$Re\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\}<\beta=\frac{2\alpha+3+\sqrt{(2\alpha+3)^2+8(\alpha+1)}}{4(\alpha+1)}. \tag{2.8}$$

Corollary 1. *$M(\alpha+1)\subset M(\alpha)$ for $\alpha>-1$.*

Proof. Noting that $1<\beta(\alpha)<\frac{\alpha+2}{\alpha+1}$, we have the corollary.

Remark 1. Theorem 1 is the improvement of Theorem A by Ganigi and Uralegaddi [1].

Taking $\alpha=0$ in Theorem 1, we have

Corollary 2. *If $f(z)\in M(1)$, Then*

$$Re\left\{-\left(\frac{zf'(z)}{f(z)}\right)\right\}>\frac{5-\sqrt{17}}{4}\qquad(z\in E), \tag{2.9}$$

that is, $f(z)$ is starlike of order $(5-\sqrt{17})/4$.

Applying Lemma 2, we prove

Theorem 2. *If $f(z)\in\Sigma$ satisfies*

$$Re\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\}<\frac{1+2(\alpha+2)(c+1)}{2(\alpha+1)(c+1)}\qquad(z\in U) \tag{2.10}$$

for $\alpha>-1$ and $c>0$, then the function $F(z)$ defined by

$$F(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \tag{2.11}$$

is in the class $M(\alpha)$, where $F(z)\neq 0$ $(z\in E)$.

Proof. From the definition of $F(z)$, it can be verified that

$$z(D^{\alpha}F(z))'=cD^{\alpha}f(z)-(c+1)D^{\alpha}F(z)\qquad(\alpha>-1). \tag{2.12}$$

Therefore, using (2.12) and

$$z(D^{\alpha}F(z))'=(\alpha+1)D^{\alpha+1}F(z)-(\alpha+2)D^{\alpha}F(z)\qquad(\alpha>-1), \tag{2.13}$$

our condition (2.10) may be written as

$$Re\left\{\frac{(\alpha+2)\frac{D^{\alpha+2}F(z)}{D^{\alpha+1}F(z)}-(\alpha+2-c)}{(\alpha+1)-(\alpha+1-c)\frac{D^{\alpha}F(z)}{D^{\alpha+1}F(z)}}\right\}<\frac{1+2(\alpha+2)(c+1)}{2(\alpha+1)(c+1)}. \tag{2.14}$$

We define the function $w(z)$ by

$$\frac{D^{\alpha+1}F(z)}{D^{\alpha}F(z)} = \frac{\alpha+2}{\alpha+1} - \frac{1}{\alpha+1}\frac{1-w(z)}{1+w(z)} = \frac{(\alpha+1)+(\alpha+3)w(z)}{(\alpha+1)(1+w(z))}. \tag{2.15}$$

Then $w(z)$ is regular in U with $w(0)=0$. Making use of the logarithmic differentiations of both sides of (2.15), and simplifying we have

$$\begin{aligned}&\frac{(\alpha+2)\frac{D^{\alpha+2}F(z)}{D^{\alpha+1}F(z)} - (\alpha+2-c)}{(\alpha+1)-(\alpha+1-c)\frac{D^{\alpha}F(z)}{D^{\alpha+1}F(z)}} \\ &= \frac{(\alpha+1)+(\alpha+3)w(z)}{(\alpha+1)(1+w(z))} + \frac{2zw'(z)}{(\alpha+1)(1+w(z))(c+(c+2)w(z))}.\end{aligned} \tag{2.16}$$

Suppose that there exists a point $z_0 \in U$ such that $\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$. Then, using Lemma 2, we have

$$z_0 w'(z_0) = kw(z_0) \qquad (k \geq 1).$$

It follows from the above that

$$\begin{aligned}&Re\left\{\frac{(\alpha+2)\frac{D^{\alpha+2}F(z_0)}{D^{\alpha+1}F(z_0)} - (\alpha+2-c)}{(\alpha+1)-(\alpha+1-c)\frac{D^{\alpha}F(z_0)}{D^{\alpha+1}F(z_0)}}\right\} \\ &= Re\left\{\frac{\alpha+2}{\alpha+1} + \frac{2kw(z_0)}{(\alpha+1)(1+w(z_0))(c+(c+2)w(z_0))}\right\} \\ &= Re\left\{\frac{\alpha+2}{\alpha+1} + \frac{2kw(z_0)(c+(c+2)\overline{w(z_0)})}{(\alpha+1)(1+w(z_0))\mid c+(c+2)w(z_0)\mid^2}\right\} \\ &\geq \frac{1+2(\alpha+2)(c+1)}{2(\alpha+1)(c+1)}.\end{aligned}$$

This contradicts our condition (2.10). Thus $|w(z)| < 1$ for all $z \in U$, or

$$Re\left\{\frac{D^{\alpha+1}F(z)}{D^{\alpha}F(z)}\right\} < \frac{\alpha+2}{\alpha+1} \qquad (z \in U). \tag{2.17}$$

Remark 2. Taking $\alpha = n \in N_0$ in Theorem 2, we have the result by Ganigi and Uralegaddi [1].

Further, we have

Theorem 3. *If $f(z) \in M(\alpha)$, then the function $F(z)$ given by*

$$G(z) = \frac{\alpha+1}{z^{\alpha+2}} \int_0^z t^{\alpha+1} f(t)dt \tag{2.18}$$

is in the class $M(\alpha+1)$, where $G(z) \neq 0$ for $z \in E$.

Proof. Since

$$(\alpha+1)D^{\alpha}f(z) = (\alpha+1)D^{\alpha+1}G(z) \tag{2.19}$$

and

$$(\alpha+1)D^{\alpha+1}f(z) = (\alpha+2)D^{\alpha+2}G(z) - D^{\alpha+1}G(z), \tag{2.20}$$

we see that

$$Re\left\{\frac{\alpha+2}{\alpha+1} - \frac{\alpha+2}{\alpha+1}\frac{D^{\alpha+2}G(z)}{D^{\alpha+1}G(z)} + \frac{1}{\alpha+1}\right\} = \left\{\frac{\alpha+2}{\alpha+1} - \frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} > 0 \tag{2.21}$$

for $f(z) \in M(\alpha)$. It follows from (2.21) that $G(z) \in M(\alpha+1)$.

Remark 3. Letting $\alpha = n \in N_0$, Theorem 3 gives the corresponding result by Ganigi and Uralegaddi [1].

3. The Class $J(\alpha)$

For our class $J(\alpha)$, we prove

Theorem 4. *If $f(z) \in J(\alpha+1)$, $\alpha > -1$, then*

$$\left\{\frac{(D^{\alpha+1}f(z))'}{(D^{\alpha}f(z))'}\right\} < \beta(\alpha) \qquad (z \in U), \tag{3.1}$$

where $\beta(\alpha)$ is given by (2.2).

Proof. Defining the function $p(z)$ by

$$\frac{(D^{\alpha+1}f(z))'}{(D^{\alpha}f(z))'} = \beta + (1-\beta)p(z) \tag{3.2}$$

with $\beta = \beta(\alpha)$, we see that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in U and

$$\frac{z(D^{\alpha+1}f(z))''}{(D^{\alpha+1}f(z))'} - \frac{z(D^{\alpha}f(z))''}{(D^{\alpha}f(z))'} = \frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}. \tag{3.3}$$

Using that

$$z(D^{\alpha}f(z))'' = (\alpha+1)(D^{\alpha+1}f(z))' - (\alpha+3)(D^{\alpha}f(z))', \tag{3.4}$$

(3.3) can be written as

$$\frac{(D^{\alpha+2}f(z))'}{D^{\alpha+1}f(z))'} = \frac{1}{\alpha+2}\left\{1 + (\alpha+1)(\beta+(1-\beta)p(z)) + \frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\right\}, \tag{3.5}$$

or

$$\begin{aligned}&\frac{\alpha+3}{\alpha+2}-\frac{(D^{\alpha+2}f(z))'}{(D^{\alpha+1}f(z))'}\\ =&\frac{1}{\alpha+2}\left\{(\alpha+2)-(\alpha+1)(\beta+(1-\beta)p(z))-\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\right\}.\end{aligned} \tag{3.6}$$

Therefore, using the same manner as in Theorem 1, we conclude that $Re\{p(z)\}>0$ $(z\in U)$. Noting that $\beta>1$, (3.2) gives

$$Re\left\{\frac{(D^{\alpha+1}f(z))'}{(D^{\alpha}f(z))'}\right\}<\beta=\frac{2\alpha+3+\sqrt{(2\alpha+3)^2+8(\alpha+1)}}{4(\alpha+1)} \tag{3.7}$$

Since $1<\beta(\alpha)<\frac{\alpha+2}{\alpha+1}$, Theorem 4 gives

Corollary 3. *$J(\alpha+1)\subset J(\alpha)$ for $\alpha>-1$.*

Remark 4. Theorem 4 is the improvement of Theorem B by Uralegaddi and Ganigi [4].

Letting $\alpha=0$ in Theorem 4, we have

Corollary 4. *If $f(z)\in J(1)$, then*

$$Re\left\{-(1+\frac{zf''(z)}{f'(z)})\right\}>\frac{5-\sqrt{17}}{4} \qquad (z\in E), \tag{3.8}$$

or, $f(z)$ is convex of order $(5-\sqrt{17})/4$.

Finally, Using the smae techniques as in the proofs of Theorem 2 and Theorem 3, we have

Theorem 5. *If $f(z)\in\Sigma$ satisfies*

$$Re\left\{\frac{(D^{\alpha+1}f(z))'}{(D^{\alpha}f(z))'}\right\}<\frac{1+2(\alpha+2)(c+1)}{2(\alpha+1)(c+1)} \qquad (z\in U) \tag{3.9}$$

for $\alpha>-1$ and $c>0$, then the function $F(z)$ given by (2.11) is in the class $J(\alpha)$, where $F(z)\neq 0$ $(z\in E)$.

Remark 5. Taking $\alpha=n\in N_0$ in Theorem 5, we have the result by Uralegaddi and Ganigi [4].

Theorem 6. *If $f(z)\in J(\alpha)$, then the function $G(z)$ defined by (2.18) is in the class $J(\alpha+1)$, where $G(z)\neq 0$ for $z\in E$.*

Remark 6. For $\alpha=n\in N_0$, Theorem 6 gives the corresponding result due to Uralegaddi and Ganigi [4].

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Institute of Mathematics Academia Sinica, Taipei, Taiwan, Republic of China.

Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577, Japan.