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ON SPIRALLIKE INTEGRAL OPERATORS

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Abstract. In this paper the integral operators

$$F(z)=\left[(\beta+\gamma)/z^{\gamma}\int_0^z [f(t)]^{\beta}t^{\gamma-1}dt\right]^{1/\beta}$$

for $f(z)\in S^{\alpha}(\lambda,a,b)$ are studied. $S^{\alpha}(\lambda,a,b)$ as a subclass of the class of all spirallike functions was introduced and studied by the authors. It is shown that $F(z)$ is also in $S^{\alpha}(\lambda,a,b)$, whenever $f(z)$ is in $S^{\alpha}(\lambda,a,b)$, under certain restrictions.

1. Introduction:

In [1], we introduced the class $S^{\alpha}(\lambda,a,b)$ of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ satisfying the condition,

$$\frac{e^{i\alpha}zf'(z)/f(z)-isin\alpha-\lambda cos\alpha}{(1-\lambda)cos\alpha}-a<b \tag{1.1}$$

where $|\alpha|<\pi/2$, $\lambda\in[0,1)$, $a\in C$, $b\in R$ with $Re\,a\geq b-\lambda/(1-\lambda)$ and $|a-1|<b$. $S^{\alpha}(\lambda,a,b)$ is the subclass of the class of all α-spiral functions. By specializing the parameters in $S^{\alpha}(\lambda,a,b)$, we obtain various subclasses studied earlier. $S^{\alpha}(\lambda,\infty,\infty)$ is the class of spirallike functions of order λ and $S^{\alpha}(o,\infty,\infty)$ is the class of all α-spiral functions. Consider the following integral,

$$F(z)=\left[(\beta+\gamma)/z^{\gamma}\int_0^z [f(t)]^{\beta}t^{\gamma-1}dt\right]^{1/\beta}=I(f), \tag{1.2}$$

where $f(z)\in S^{\alpha}(\lambda,a,b),\beta$ and γ are complex numbers with $\beta=|\beta|e^{i\alpha},|\alpha|<\pi/2$. In this paper, we show that $F(z)\in S^{\alpha}(\lambda,a,b)$, under certain restrictions.

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2. Perliminaries:

It is convenient to phrase our results in terms of subordination. If f and g are analytic in the unit disc $E = \{z/|z| < 1\}$, we say that f is subordinate to g, written $f(z) \prec g(z)$ or $f \prec g$, if g is univalent, $f(0) = g(0)$ and $f(E) \subset g(E)$. Using this concept, the functions in $S^{\alpha}(\lambda, a, b)$ can be redefined. From (8) of [1], we can write

$$zf'(z)/f(z) \prec \frac{be^{i\alpha} + [c(1-\lambda)cos\alpha + (1-\overline{a})e^{i\alpha}]z}{be^{i\alpha} + (1-\overline{a})e^{i\alpha}]z} \tag{2.1}$$

or equivalently,

$$e^{i\alpha} zf'(z)/f(z) \prec \frac{e^{i\alpha} + Az}{1 + Bz} \tag{2.2}$$

where $f(z) \in S^{\alpha}(\lambda, a, b), A = \{c(1-\lambda)cos\alpha + (1-\overline{a})e^{i\alpha}\}/b, B = (1-\overline{a})/b, c = b^2 - |1 - \overline{a}|^2$ and α, λ, a, b are as above.

We need the following lemmas to prove our main theorem.

Lemma 1 [3,Corollary 1.1]: *Let β and γ be complex numbers with $\beta \neq 0$ and let $f(z) = z + a_2 z^2 + \ldots$, be regular in E. If*

$$Re\{\beta zf'(z)/f(z) + \gamma\} > 0$$

for $z \in E$, then the function F defined by

$$F(z) = \left[(\beta+\gamma)/z^{\gamma} \int_0^z [f(t)]^{\beta} t^{\gamma-1} dt\right]^{1/\beta} = I(f)(z) \tag{2.3}$$

is regular in $E, F(z)/z \neq 0$ and $Re\{\beta zF'(z)/F(z) + \gamma\} > 0$.

Lemma 2 [3, Corollary 3.1]: *Let β and γ be complex numbers with $\beta \neq 0$, and let $f(z) = z + a_2 z^2 + \ldots$, and $g(z) = z + b_2 z^2 + \ldots$, be regular functions in E that satisfy*

$$zf'(z)/f(z) \prec zg'(z)/g(z) \tag{2.4}$$

Let $P(z) = \beta\{zg'(z)/g(z)\} + \gamma$ and suppose P satisfies

(i) $ReP(z) > 0$

and

(ii) *P and $1/P$ are convex (univalent) in E.*

If $E \equiv I(f)$ and $G \equiv I(g)$, where I is given by (2.3) then F and G are regular in $E, zG'(z)/G(z)$ is univalent in E and

$$zF'(z)/F(z) \prec zG'(z)/G(z) \prec zg'(z)/g(z) \tag{2.5}$$

Note that the condition (i) of Lemma 2 and Lemma 1 imply that G is regular in E with $G(z)/z \neq 0$. Also from (i) and (2.4), we have $Re\{\beta[zf'(z)/f(z)] + \gamma\} > 0$. Which together with Lemma 1 implies that F is regular in E and $F(z)/z \neq 0$.

3. Now we prove our main theorem.

Theorem 1 : *Let β and γ be complex numbers with $Re\beta > 0$, satisfying conditions,*

$$Re\{\overline{B}\beta Ae^{-i\alpha} + |B|^2\gamma + \beta Ae^{-i\alpha} - \beta B\} > 0 \tag{3.1}$$

and

$$|\beta Ae^{-i\alpha} + B\gamma| < |\beta + \gamma| \tag{3.2}$$

where $A = \{c(1-\lambda)cos\alpha + (1-\overline{a})e^{i\alpha}/b, B = (1-\overline{a})/b, c = b^2 - |1-\overline{a}|^2$. If $\beta = |\beta|e^{i\alpha}$ and the functions $f(z) = z + a_2z^2 + \ldots, g(z) = z + b_2z^2 + \ldots$ are regular in E satisfying

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \frac{e^{i\alpha} + Az}{1 + Bz} \equiv e^{i\alpha}\frac{zg'(z)}{g(z)} \tag{3.3}$$

Then the functions

$$F(z) = I(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z \{f(t)\}^\beta \cdot t^{\gamma-1} \cdot dt\right]^{1/\beta} \tag{3.4}$$

and $G(z) = I(g)(z)$ are regular in E and satisfy

$$e^{i\alpha}\frac{zF'(z)}{F(z)} \prec e^{i\alpha}\frac{zG'(z)}{G(z)} \prec e^{i\alpha}\frac{zg'(z)}{g(z)} \tag{3.5}$$

Proof. The function g defined by (3.3) is given by

$$g(z) = z(1 + Bz)^{(A/Be^{-i\alpha}-1} = z + b_2z^2 + \ldots$$

(all powers are principal powers) and satisfies

$$zf'(z)/f(z) \prec zg'(z)/g(z)$$

Now let

$$P(z) = \beta\left\{\frac{zg'(z)}{g(z)}\right\} + \gamma \tag{3.6}$$

$$= \frac{(\beta+\gamma) + (\beta Ae^{-i\alpha} + B\gamma)z}{1 + Bz} \tag{3.7}$$

Putting $(\beta Ae^{-i\alpha} + B\gamma)/(\beta + \gamma) = \rho$, we get,

$$P(z) = \{(\beta + \gamma)(1 + \rho z)\}/(1 + Bz) \tag{3.8}$$

(3.7) is a linear transformation which maps $|z| < 1$ on to a circle with centre at

$$\frac{(\beta+\gamma)-\overline{B}(\beta Ae^{-i\alpha}+B\gamma)}{(1-|B|^2)} \quad \text{and}$$

radius as

$$|\beta Ae^{-i\alpha}+B\gamma-(\beta+\gamma)\overline{B}|/(1-|B|^2),$$

which implies that $ReP(z) \geq 0$ if and only if (3.1) is satisfied. $P(z)$ and $1/P(z)$ are convex if $|B| < 1$ and (3.2) is satisfied. Thus P satisfies all the conditions of Lemma 2, and (3.5) is proved.

Remark. (1) For $a = b = \infty$, the condition (3.1) reduces to the condition (5) of Theorem 1 of [2].

(2) (3.5) essentially proves G is the best dominant over the class of integral operators of functions in $S^\alpha(\lambda, a, b)$ and is given by

$$\frac{zG'(z)}{G(z)} = \frac{1}{\beta}\left[\frac{z^{\beta+\gamma}(1+Bz)^{(\frac{\beta Ae^{-i\alpha}}{\beta}-\beta}}{\int_0^z t^{\beta+\gamma-1}(1+Bt)^{(\{\beta Ae^{-i\alpha}/B\}-\beta)}dt} - \gamma\right]$$

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