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UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

B.A. URALEGADDI*, M.D. GANIGI** AND S.M. SARANGI*

Abstract. Coefficient inequalities, distortion and covering Theorems and extreme points are determined for univalent functions with positive coefficients.

1. Introduction

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$, A function $f \in S$ is said to be starlike of order $\alpha, 0 \leq \alpha < 1$, denoted by $f \in S^*(\alpha)$, if $Re\, zf'(z)/f(z) > \alpha$ for $z \in E$ and is said to be convex of order $\alpha, 0 \leq \alpha < 1$, denoted by $f \in K(\alpha)$, if $Re(1 + zf''(z)/f'(z)) > \alpha$ for $z \in E$. $S^*(0) = S^*$ and $K(0) = K$ are respectively the classes of starlike and convex functions in S.

For $1 < \beta \leq 4/3$ and $z \in E$, let $M(\beta) = \{f \in S : Re\, zf'(z)/f(z) < \beta\}$ and $L(\beta) = \{f \in S : Re(1 + zf''(z)/f'(z)) < \beta\}$. Further let V be the subclass of S consisting of functions of the form $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$.

Let $V^*(\alpha) = S^*(\alpha) \cap V, V_K(\alpha) = K(\alpha) \cap V$ and $V(\beta) = M(\beta) \cap V, U(\beta) = L(\beta) \cap V$. $V^*(0) = V^*$ and $V_K(0) = V_K$ are respectively the classes of starlike and convex functions in V.

In this paper coefficient in equalities, distortion and covering Theorems and extreme points are determined for classes $V(\beta)$ and $U(\beta)$. Further order of starlikeness and convexity are obtained for the classes $V(\beta)$ and $U(\beta)$ respectively.

In [2] *H*. Silverman has studied the univalent functions with negative coefficients.

2. Coefficient inequalities.

Theorem 2.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* S. *If* $\sum_{n=2}^{\infty} (n - \beta)|a_n| \leq \beta - 1$ *then* $f \in M(\beta)$.

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Proof. Let $\sum_{n=2}^{\infty}(n-\beta)|a_n| \le \beta - 1$. It sufficies to show that

$$\left|\frac{zf'(z)/f(z)-1}{zf'(z)/f(z)-(2\beta-1)}\right| < 1, \quad z \in E.$$

We have

$$\begin{aligned}
&\left|\frac{zf'(z)/f(z)-1}{zf'(z)/f(z)-(2\beta-1)}\right| \\
&\le \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{2(\beta-1)-\sum_{n=2}^{\infty}(n-2\beta+1)|a_n||z|^{n-1}} \\
&\le \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{2(\beta-1)-\sum_{n=2}^{\infty}(n-2\beta+1)|a_n|}
\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}(n-1)|a_n| \le 2(\beta-1) - \sum_{n=2}^{\infty}(n-2\beta+1)|a_n|$$

which is equivalent to

$$\sum_{n=2}^{\infty}(n-\beta)|a_n| \le \beta - 1. \tag{2.1}$$

But (2.1) is true by hypothesis. Hence

$$\left|\frac{zf'(z)/f(z)-1}{zf'(z)/f(z)-(2\beta-1)}\right| < 1, \quad z \in E$$

and the theorem is proved.

Corollary 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in S. If $\sum_{n=2}^{\infty} n(n-\beta)|a_n| \le \beta - 1$ then $f \in L(\beta)$.*

Proof. Since $f \in L(\beta)$ if and only if $zf' \in M(\beta)$, the result follows.

For functions in $V(\beta)$ the converse of Theorem 2.1 is also true.

Theorem 2.3. *A function $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ is in $V(\beta)$ if and only if $\sum_{n=2}^{\infty}(n-\beta)|a_n| \le \beta - 1$.*

Proof. In view of Theorem 2.1, it suffices to show the only if part. Suppose

$$Re\, zf'(z)/f(z) = Re\, \frac{z + \sum_{n=2}^{\infty} n|a_n|z^n}{z + \sum_{n=2}^{\infty} |a_n|z^n} < \beta, \quad z \in E. \tag{2.2}$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through real values we obtain $1 + \sum_{n=2}^{\infty} n|a_n| \le \beta(1 + \sum_{n=2}^{\infty} |a_n|)$. Thus we have $\sum_{n=2}^{\infty}(n-\beta)|a_n| \le \beta - 1$, and the proof is complete.

Corollary 2.4. *A function $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n$ is in $U(\beta)$ if and only if $\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leq \beta - 1$.*

Proof. The proof follows as that of Corollary 2.2.

Remark. The above corollary is true even if $1 < \beta \leq 3/2$.

3. Distortion and Covering Theorems

Theorem 2.3 enables us to prove the following

Theorem 3.1 *If $f \in V(\beta)$ then*

$$r - \frac{\beta - 1}{2 - \beta} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2 - \beta} r^2 \qquad (|z| = r)$$

with equality for $f(z) = z + \frac{\beta-1}{2-\beta} z^2 \qquad (z = \pm r)$

Proof. From Theorem 2.3, we have

$$(2-\beta) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} (n-\beta)|a_n| \leq \beta - 1. \quad \text{Thus}$$

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{\beta - 1}{2 - \beta} r^2.$$

Similarly

$$|f(z)| \geq r - \sum_{n=2}^{\infty} |a_n| r^n \geq r - r^2 \sum_{n=2}^{\infty} |a_n| \geq r - \frac{\beta - 1}{2 - \beta} r^2$$

Corollary 3.2. *If $f \in U(\beta)$ then*

$$r - \frac{\beta - 1}{2(2 - \beta)} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2(2 - \beta)} r^2 \qquad (|z| = r)$$

with equality for $f(z) = z + \frac{\beta-1}{2(2-\beta)} z^2 \qquad (z = \pm r)$

Theorem 3.3. *The disk $|z| < 1$ is mapped on to a domain that contains the disk $|w| < (3 - 2\beta)/(2 - \beta)$ by any $f \in V(\beta)$ and on to a domain that contains the disk $|w| < (5 - 3\beta)/2(2 - \beta)$ by any $f \in U(\beta)$. The theorem is sharp for the extremal functions $z + \frac{\beta-1}{2-\beta} z^2 \in V(\beta)$ and $z + \frac{\beta-1}{2(2-\beta)} z^2 \in U(\beta)$.*

Proof. By letting $r \to 1$ in Theorem 3.1 and Corollary 3.2 the results are obtained.

Theorem 3.4. *If $f \in V(\beta)$ then*

$$1 - \frac{2(\beta - 1)}{2 - \beta} r \leq |f'(z)| \leq 1 + \frac{2(\beta - 1)}{2 - \beta} r \qquad (|z| = r)$$

with equality for $f(z) = z + \frac{\beta-1}{2-\beta}z^2 \qquad (z = \pm r)$

Proof. We have

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \leq 1 + r\sum_{n=2}^{\infty} n|a_n| \tag{3.1}$$

In view of Theorem 2.3 we have

$$\sum_{n=2}^{\infty} n|a_n| \leq \beta - 1 + \beta\sum_{n=2}^{\infty} |a_n| \leq \beta - 1 + \frac{\beta(\beta-1)}{2-\beta} = \frac{2(\beta-1)}{2-\beta} \tag{3.2}$$

From (3.1) and (3.2) it follows that $|f'(z)| \leq 1 + \frac{2(\beta-1)}{2-\beta}r$.

Similarly

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \geq 1 - r\sum_{n=2}^{\infty} n|a_n| \geq 1 - \frac{2(\beta-1)}{2-\beta}r.$$

This completes the proof.

Corollary 3.5. *If* $f \in U(\beta)$ *then*

$$1 - \frac{\beta-1}{2-\beta}r \leq |f'(z)| \leq 1 + \frac{\beta-1}{2-\beta}r \qquad (|z| = r).$$

Equality holds for $f(z) = z + \frac{\beta-1}{2(2-\beta)}z^2 \qquad (z = \pm r)$

4. Order of Starlikeness and Convexity

Theorem 4.1. *If* $f \in V(\beta)$ *then* $f \in V^*((4-3\beta)/(3-2\beta))$

Proof. Since $\sum_{n=2}^{\infty} |a_n|(n-\alpha)/(1-\alpha) \leq 1$ [2] is a sufficient condition for $f \in S$ to be in $S^*(\alpha)$, in view of Theorem 2.3 we must prove that

$$\sum_{n=2}^{\infty} \frac{(n-\beta)}{\beta-1}|a_n| \leq 1 \text{ implies } \sum_{n=2}^{\infty} \frac{n-(4-3\beta)/(3-2\beta)}{1-(4-3\beta)/(3-2\beta)}|a_n| \leq 1.$$

It suffices to show that

$$\frac{n-\beta}{\beta-1} \geq \frac{n-(4-3\beta)/(3-2\beta)}{1-(4-3\beta)/(3-2\beta)} = \frac{(3-2\beta)n - 4 + 3\beta}{\beta-1}, n = 2, 3, \ldots \tag{4.1}$$

But (4.1) is equivalent to $(\beta-1)(n-2) \geq 0, n = 2, 3, \ldots$ and the theorem is proved.

Corollary 4.2. $V(\beta) \subset V(4/3) \subset V^*$.

Thus all functions in $V(\beta)$ are starlike. There is no converse to Theorem 4.1. That is a function in $V^*(\alpha)$ need not have $Re\, zf'(z)/f(z) < \beta$. To show this we need only to find the coefficients $|a_n|$ for which

$$\sum_{n=2}^{\infty} n|a_n| \leq 1 \text{ and } \sum_{n=2}^{\infty}(3n-4)|a_n| > 1. \tag{4.2}$$

Note that the function $f(z) = z + z^2/6 + z^3/6$ satisfies both inequalities in (4.2).

Corollary 4.3. *If $f \in U(\beta)$ then $f \in V_K((4-3\beta)/(3-2\beta))$*

Corollary 4.4. $U(\beta) \subset U(4/3) \subset V_K$.

The above corollary is comparable to the following results of *S.* Ozaki [1] and R.Singh and S.Singh [3], for wider class of functions.

Theorem A. [1]. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in E and satisfies $Re\,(1 + zf''(z)/f'(z)) < 3/2$ then f is univalent in E.*

Theorem B. [3]. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytice in E and satisfies $Re(1 + zf''(z)/f'(z)) < 3/2$ then f is starlike in E.*

Theorem 4.5. *If $f \in U(\beta)$ then $f \in V(2/(3-\beta))$.*
Proof is similar to that of Theorem 4.1.
Putting $\beta = 4/3$ in Theorem 4.5 we have

Corollary 4.6. $U(4/3) \subset V(6/5)$.

From Corollary 4.6 and Theorem 4.1, we have

Corollary 4.7. $U(4/3) \subset V^*(2/3)$.

Since Theorem 4.5 is true even if $1 < \beta \leq 3/2$ the following Corollary is obtained.

Corollary 4.8. *If $f(z) = z + \sum_{n=2}^{\infty} |a_n| z^n \in V$, satisfies $Re(1 + zf''(z)/f'(z)) < 3/2$ then $Re\, zf'(z)/f(z) < 4/3$ i.e. $f \in V(4/3)$.*

5. Extreme Points

In view of Theorem 2.3 the class $V(\beta)$ is closed under convex linear combinations. We shall determine the extreme points of $V(\beta)$.

Theorem 5.1. *Let $f_1(z) = z$ and $f_n(z) = z + \frac{\beta-1}{n-\beta} z^n, n = 2, 3, \ldots$. Then $f \in V(\beta)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$.*
Proof is similar to that of Theorem 9 in [2].

Corollary 5.2. *The extreme points of $V(\beta)$ are the functions $f_n(z), n = 1, 2, \ldots$*

Corollary 5.3. *The extreme points of* $U(\beta)$ *are the functions* $f_1(z) = z$ *and* $f_n(z) = z + \frac{\beta-1}{n(n-\beta)} z^n, n = 2, 3, \ldots$

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* Department of Mathematics, Karnatak University, Dharwad-580 003, Karnataka, India.

** Department of Mathematics, Karnatak Arts College, Dharwad-580 001, Karnataka, India.