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RELATION ON SOME SUMMABILITY METHODS

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Abstract. In this note a new theorem concerning $\varphi - |C,\alpha|_k$ summability of infinite series is proved. This Theorem contains as a special case the result of Bor (1986) which gives a relation between the two summability methods $|\overline{N},p_n|_k$ and $|C,1|_k$.

1. Introduction

Let $A=(a_{nk})$ be an infinite matrix of complex numbers $a_{nk}(n,k=1,2,\cdots)$ and let (φ_n) be a sequence of complex numbers. Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). We denote by $A_n(s)$ the A-transform of the sequence $s=(s_r)$,

$$A_n(s)=\sum_{r=1}^{\infty} a_{nr}s_r. \tag{1.1}$$

We say that the series $\sum a_n$ is summable $|A|$, if

$$\sum_{n=1}^{\infty} |A_n(s)-A_{n-1}(s)|<\infty. \tag{1.2}$$

and it is said to be summable $\varphi-|A|_k,\ k\geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} |\varphi_n[A_n(s)-A_{n-1}(s)]|^k<\infty. \tag{1.3}$$

If we take $\varphi_n=n^{1-1/k}$ (resp. $\varphi_n=n^{\delta+1-1/k}$, where $\delta\geq 0$), then $\varphi-|A|_k$ summability is the same as $|A|_k$ (resp. $|A,\delta|_k$) summability (see [4], [5]).

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Let σ_n^δ and η_n^δ denote the n-th Cesàro mean of order $\delta(\delta > -1)$ of the sequences (s_n) and (na_n) respectively. The series $\sum a_n$ is said to be absolutely summable (C, δ) with index k, or simply summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} \frac{1}{n} |\eta_n^\delta|^k < \infty.$$

Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty (P_{-1} = p_{-1} = 0).$$

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{k-1} |T_n - T_{n-1}|^k < \infty \quad \text{(see [2])}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v.$$

If we take $p_n = 1$, then $|\overline{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. $|\overline{N}, p_n|_1$ is the same as $|\overline{N}, p_n|$. In general the two methods $|C, \delta|_k$ and $|\overline{N}, p_n|_k$ are not comparable.

Bor (1986) established the following result.

Theorem A. *Let (p_n) be a sequence of positive real constants such that as $n \to \infty$*

$$(i)\ np_n = O(P_n), \qquad (ii)\ P_n = O(np_n). \tag{I}$$

If $\sum a_n$ is summable $|\overline{N}, p_n|_k$, then it is summable $|C, 1|_k$, $k \geq 1$.

2. Main Result

We prove the following

Theorem B. *Let (χ_n) be a positive non-decreasing sequence and let (β_n) and (λ_n) be sequences such that*

$$|\Delta \lambda_n| \leq \beta_n \tag{2.1}$$

$$\beta_n \to 0 \quad as\ n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|\chi_n < \infty \tag{2.3}$$

$$|\lambda_n|\chi_n = O(1). \tag{2.4}$$

If there exists an $\epsilon > 0$ *such that the sequence* $(n^{\epsilon-k}|\varphi_n|^k)$ *is non-increasing and*

$$\sum_{v=1}^{n} v^{1-k\alpha}(\frac{P_v}{p_v})^{k-1}|\varphi_v \Delta T_{v-1}|^k = O(\chi_n), \ n \to \infty \tag{2.5}$$

where (p_n) *is a sequence of positive real constants satisfying (I), and* T_n *is the* $(\overline{N}, p_n)$*-mean of the series* $\sum a_n$*, then the series* $\sum a_n\lambda_n$ *is summable* $\varphi - |C,\alpha|_k$, $k \geq 1$, $1 - 1/k < \alpha \leq 1$.

Remark. If we put $\alpha = 1, \lambda_n = 1, \varphi_v = v^{1-1/k}$, and $\chi_n = 1$ in Theorem B, we obtain Theorem A.

3. Lemmas

Lemma 1[6]. *If the conditions* (2.1)-(2.4) *are satisfied, then*

$$n\beta_n\chi_n = O(1), \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} \beta_n\chi_n < \infty \tag{3.2}$$

Lemma 2[7]. *If* $\sigma > \delta > 0$, *then*

$$\sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^{\sigma}} = O(v^{\delta-\sigma}), \quad m \to \infty \tag{3.3}$$

4. Proof of Theorem B

Let t_n^α be the n-th (C,α)-mean of the sequence $(na_n\lambda_n)$. Then in order to prove the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^k}|\varphi_n t_n^\alpha|^k < \infty, \tag{4.1}$$

where

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v \ a_v\lambda_v,$$

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

As

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v,$$

then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v.$$

Abel's transformation gives

$$\begin{aligned}
t_n^\alpha =& \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} P_{v-1}a_v\left\{vA_{n-v}^{\alpha-1}P_{v-1}^{-1}\lambda_v\right\}\\
=& \frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}P_{r-1}a_r\right)\Delta\left(vA_{n-v}^{\alpha-1}P_{v-1}^{-1}\lambda_v\right) + \left(\sum_{r=1}^{n}P_{r-1}a_r\right)nP_{v-1}^{-1}\lambda_n\right]\\
=& \frac{1}{A_n^\alpha}\left[\sum_{v=1}^{n-1}\left\{v\frac{P_v}{p_v}\Delta A_{n-v}^{\alpha-1}\lambda_v\Delta T_{v-1} + vA_{n-v-1}^{\alpha-1}\lambda_v\Delta T_{v-1} - \frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\lambda_v\Delta T_{v-1}\right.\right.\\
&+ \left.\left.(v+1)\frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\Delta\lambda_v\Delta T_{v-1}\right\} + n\frac{P_n}{p_n}\lambda_n\Delta T_{n-1}\right]\\
=& t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha + t_{n,5}^\alpha, \text{ say.}
\end{aligned}$$

In order to prove the Theorem, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^k}|\varphi_n t_{n,j}^\alpha|^k < \infty, \quad j = 1,2,3,4,5.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
&\sum_{n=2}^{m+1}\frac{1}{n^k}|\varphi_n t_{n,1}^\alpha|^k\\
\le& \sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^k(A_n^\alpha)^k}\sum_{v=1}^{n-1}v^k\left(\frac{P_v}{p_v}\right)^k|\Delta A_{n-v}^{\alpha-1}|\,|\lambda_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}|\Delta A_{n-v}^{\alpha-1}|\right\}^{k-1}\\
=& O(1)\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^{k+k\alpha}}\sum_{v=1}^{n-1}v^k\left(\frac{P_v}{p_v}\right)^k(n-v)^{\alpha-2}|\lambda_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\right\}^{k-1}\\
=& O(1)\sum_{v=1}^{m}v^k\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k(n-v)^{\alpha-2}}{n^{k+k\alpha}}, 0<\alpha<1,\\
&(\text{when } \alpha = 1,\ t_{n,1}^\alpha = 0,\ \text{as } \Delta A_{n-v}^{\alpha-1} = 0)\\
=& O(1)\sum_{v=1}^{m}v^{-k\alpha}\left(\frac{P_v}{p_v}\right)^k|\lambda_v|^k|\varphi_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{1}{(n-v)^{2-\alpha}}
\end{aligned}$$

$$=O(1)\sum_{v=1}^{m}|\lambda_v|v^{1-k\alpha}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta T_{v-1}|^k$$
$$=O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\chi_v+O(1)|\lambda_m|\chi_m$$
$$=O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|\chi_v+O(1)|\lambda_m|\chi_m$$
$$=O(1)\sum_{v=1}^{m}\beta_v\chi_v+O(1)|\lambda_m|\chi_m$$
$$=O(1).$$

In view of (2.1), (2.4), (2.5), (3.2), (3.3) and the boundedness of λ_n (see [3])

$$\sum_{n=2}^{m+1}\frac{1}{n^k}|\varphi_n t_{n,2}^{\alpha}|^k\leq\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^kA_n^{\alpha}}\sum_{v=1}^{n-1}v^kA_{n-v-1}^{\alpha-1}|\lambda_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}\frac{A_{n-v-1}^{\alpha-1}}{A_n^{\alpha}}\right\}^{k-1}$$
$$=O(1)\sum_{v=1}^{m}v^k|\lambda_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^kA_{n-v-1}^{\alpha-1}}{n^kA_n^{\alpha}}$$
$$=O(1)\sum_{v=1}^{m}v^k|\lambda_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k(n-v)^{\alpha-1}}{n^{k+\alpha}}$$
$$=O(1)\sum_{v=1}^{m}v^{\epsilon}|\lambda_v|^k|\varphi_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{\alpha+\epsilon}}$$
$$=O(1)\sum_{v=1}^{m}|\lambda_v|^k|\varphi_v|^k|\Delta T_{v-1}|^k$$
$$=O(1)\sum_{v=1}^{m}|\lambda_v|v^{1-k}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta T_{v-1}|^k$$
$$=O(1)\sum_{v=1}^{m}|\lambda_v|v^{1-k\alpha}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta T_{v-1}|^k$$
$$=O(1),\ \text{as in the case of } t_{n,1}^{\alpha}$$

$$\sum_{n=2}^{m+1}\frac{1}{n^k}|\varphi_n t_{n,3}^{\alpha}|^k\leq\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n^k(A_n^{\alpha})}\sum_{v=1}^{n-1}(\frac{P_v}{p_v})^kA_{n-v-1}^{\alpha-1}|\lambda_v|^k|\Delta T_{v-1}|^k\left\{\sum_{v=1}^{n-1}\frac{A_{n-v-1}^{\alpha-1}}{A_n^{\alpha}}\right\}^{k-1}$$
$$=O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^k|\lambda_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k|A_{n-v-1}^{\alpha-1}}{n^kA_n^{\alpha}}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{\epsilon-k}(\frac{P_v}{p_v})^k|\lambda_v|^k|\varphi_v|^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-1}}{n^{\alpha+\epsilon}}\\
&= O(1)\sum_{v=1}^{m} v^{-k}(\frac{P_v}{p_v})^k|\lambda_v|^k|\varphi_v|^k|\Delta T_{v-1}|^k\\
&= O(1)\sum_{v=1}^{m} |\lambda_v| v^{1-k}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta_{v-1}|^k\\
&= O(1), \text{ as in the case of } t_{n,2}^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\frac{1}{n^k}|\varphi_n t_{n,4}^{\alpha}|^k\\
\le &\sum_{n=2}^{m+1}\frac{|\varphi_n|^k}{n(A_n^{\alpha})^k}\sum_{v=1}^{n-1}(v+1)(\frac{P_v}{p_v})^k(A_{n-v-1}^{\alpha-1})^k|\Delta\lambda_v|\,|\Delta T_{v-1}|^k\left\{\frac{1}{n}\sum_{v=1}^{n-1}(v+1)|\Delta\lambda_v|\right\}^{k-1}\\
= &O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^k|\Delta\lambda_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k|(A_{n-v-1}^{\alpha-1})^k}{n(A_n^{\alpha})^k}\\
= &O(1)\sum_{v=1}^{m} v(\frac{P_v}{P_v})^k|\Delta\lambda_v||\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{|\varphi_n|^k(n-v)^{k\alpha-k}}{n^{1+k\alpha}}\\
= &O(1)\sum_{v=1}^{m} v^{1+\epsilon-k}(\frac{P_v}{p_v})^k|\Delta\lambda_v||\varphi_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{k\alpha-k}}{n^{1+\epsilon+k\alpha-k}}\\
= &O(1)\sum_{v=1}^{m} v^{1-k}(\frac{P_v}{p_v})^k|\Delta\lambda_v||\varphi_v|^k|\Delta T_{v-1}|^k\\
= &O(1)\sum_{v=1}^{m}|\Delta\lambda_v|v^{2-k}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta T_{v-1}|^k\\
= &O(1)\sum_{v=1}^{m} v\beta_v v^{1-k\alpha}(\frac{P_v}{p_v})^{k-1}|\varphi_v\Delta T_{v-1}|^k\\
= &O(1)\sum_{v=1}^{m}|\Delta(v\beta_v)|\chi_v + O(1)m\beta_m\chi_m\\
= &O(1)\sum_{v=1}^{m}\beta_v\chi_v + \sum_{v=1}^{m}(v+1)|\Delta\beta_v|\chi_v + O(1)m\beta_m\chi_m\\
= &O(1),
\end{aligned}
$$

in veiw of (2.3), (2.5), (3.1), (3.2) and (3.3).

$$\sum_{n=1}^{m} \frac{1}{n^k} |\varphi_n t_{n,5}^{\alpha}|^k \leq \sum_{n=1}^{m} \frac{1}{(A_n^{\alpha})^k} |\lambda_n|^k (\frac{P_n}{p_n})^k |\varphi_n \Delta T_{n-1}|^k$$
$$= O(1) \sum_{n=1}^{m} |\lambda_n| n^{1-k\alpha} (\frac{P_n}{p_n})^{k-1} |\varphi_n \Delta T_{n-1}|^k$$
$$= O(1).$$

This completes the proof of the Theorem.

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