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CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k \qquad (a_k \geq 0, \quad a_{-1} > 0)$$

which are analytic in the annulus $D = \{z \mid 0 < |z| < 1\}$. Let $\Sigma_{p,1}$ and $\Sigma_{p,2}$ denote subclasses of Σ_p satisfying $f(z_0) = \frac{1}{z_0}$ and $f'(z_0) = -\frac{1}{z_0^2}$ $(-1 < z_0 < 1,\ z_0 \neq 0)$, respectively. Properties of certain subclasses of $\Sigma_{p,1}$ and $\Sigma_{p,2}$ are investigated and sharp results are obtained. Also a new characterization for certain subclass of Σ_p is proved.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k \ (a_{-1} \neq 0) \tag{1.1}$$

which are analytic in the annulus $D = \{z : 0 < |z| < 1\}$. The Hadamard product or convolution of two functions f, g in Σ will denoted by $f * g$. Let

$$D^n f(z) = \frac{1}{z(1-z)^{n+1}} * f(z) \ (z \in D, n \in N_0 = \{0, 1, 2, \cdots\}). \tag{1.2}$$

Uralegaddi and Ganigi [3] observed that

$$D^n f(z) = \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} \quad (z \in D,\ n \in N_0). \tag{1.3}$$

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Also, we note that $D^0 f = f$.

Let Σ_p be the subclass of Σ consisting functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k \ (a_k \geq 0,\ a_{-1} > 0). \tag{1.4}$$

Let $\sum_p(n, A, B, \alpha)$ denote the class of functions $f \in \sum_p$ such that

$$\frac{z(D^n f(z))'}{D^n f(z)} = -\frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)} \tag{1.5}$$

$$(-1 \leq A < B \leq 1,\ A + B \geq 0,\ 0 \leq \alpha < 1,\ n \in N_0),$$

where $z \in U = \{z : |z| < 1\}$ and $w \in H = \{w$ analytic in U, $w(0) = 0$ and $|w(z)| < 1, z \in U\}$.

For a given real number $z_0(-1 < z_0 < 1, z_0 \neq 0)$, let $\Sigma_{p,1}$ and $\Sigma_{p,2}$ be the subclasses of Σ_p satisfying $f(z_0) = \frac{1}{z_0}$ and $f'(z_0) = -\frac{1}{z_0^2}$, respectively.

Let $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ and $\Sigma_{p,2}(z_0, n, A, B, \alpha)$ be the subclasses of Σ_p defined as follows:

$$\Sigma_{p,i}(z_0, n, A, B, \alpha) = \Sigma_p(n, A, B, \alpha) \cap \Sigma_{p,i}\ (i = 1, 2). \tag{1.6}$$

In this paper we obtain necessary and sufficient conditions for functions to be in $\Sigma_p(n, A, B, \alpha)$ and $\Sigma_{p,i}(z_0, n, A, B, \alpha)$ $(i = 1, 2)$. We determine extreme points and radius of conexity for the classes $\Sigma_{p,i}(z_0, n, A, B, \alpha)$ $(i = 1, 2)$. Also closure theorems are proved for these subclasses. Further a new characterization theorem is proved for the class $\Sigma_p(n, A, B, \alpha)$. Techniques used are similar to those of Silverman [4].

Remarks. (1). Taking $n = 0, \alpha = 0$ and $a_{-1} = 1$ in the class $\sum_p(n, A, B, \alpha)$, we can obtain the results studied by Cho [2].

(2). Taking $n = 0, A = -\beta, B = \beta(0 < \beta \leq 1)$ and $a_{-1} = 1$ in the class $\Sigma_p(n, A, B, \alpha)$, we can get the results studied by Mogra, Reddy and Juneja [1].

2. The Main Results

We now introduce the following notations for brevity:

$$D_k = (n + k + 1)![(k + 1)(B + 1) + (A - B)(1 - \alpha)],$$

$$E_k = n!(k + 1)!(B - A)(1 - \alpha),\ F_k = D_k + E_k z_0^{k+1}.$$

Theorem 2.1. *A function $f \in \Sigma_p$ is in $\Sigma_p(n, A, B, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{D_k a_k}{E_k} \leq a_{-1}. \tag{2.1}$$

Proof. Suppose $f \in \Sigma_p(n, A, B, \alpha)$. Then

$$\frac{z(D^n f(z))'}{D^n f(z)} = -\frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1 + Bw(z)} \tag{2.2}$$

$$(-1 \leq A < B \leq 1, A + B \geq 0, w(z) \in H, 0 \leq \alpha < 1, z \in U).$$

From (2.2), we get

$$w(z) = -\frac{z(D^n f(z))' + D^n f(z)}{Bz(D^n f(z))' + [B + (A-B)(1-\alpha)]D^n f(z)} \tag{2.3}$$

and $|w(z)| < 1$ implies

$$|w(z)| = \left| \frac{\sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}(k+1)a_k z^k}{(B-A)(1-\alpha)\frac{a_{-1}}{z} - \sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}[(k+1)B + (A-B)(1-\alpha)]a_k z^k} \right| < 1. \tag{2.4}$$

Since $|Re(z)| \leq |z|$, we have, from (2.4),

$$Re\left(\frac{\sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}(k+1)a_k z^{k+1}}{(B-A)(1-\alpha)a_{-1} - \sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}[(k+1)B + (A-B)(1-\alpha)]a_k z^{k+1}} \right) < 1. \tag{2.5}$$

We consider real values of z and take $z = r$ with $0 \leq r < 1$. Then, for sufficiently small r, the denominator of (2.5) is positive and so it is positive for all r with $0 \leq r < 1$, since $w(z)$ is analytic for $|z| < 1$. Then (2.5) gives

$$\sum_{k=1}^{\infty} \frac{D_k a_k r^{k+1}}{E_k} < a_{-1}. \tag{2.6}$$

Letting $r \to 1$, we get (2.1).

Conversely, suppose $f \in \Sigma_p$ and f satisfies (2.1). For $|z| = r, 0 \leq r < 1$, (2.6) is implied by (2.1), since $r^{k+1} < 1$. So we have

$$\left| \sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}(k+1)a_k z^{k+1} \right| \leq \sum_{k=1}^{\infty} \frac{(n+k+1)!}{n!(k+1)!}(k+1)a_k r^{k+1} \tag{2.7}$$

$$< (B-A)(1-\alpha)a_{-1} - \sum_{k=1}^{\infty} \frac{(n+k+1)![(k+1)B + (A-B)(1-\alpha)]}{n!(k+1)!} a_k r^{k+1}$$

$$\leq \left| (B-A)(1-\alpha)a_{-1} - \sum_{k=1}^{\infty} \frac{(n+k+1)![(k+1)B + (A-B)(1-\alpha)]}{n!(k+1)!} a_k z^{k+1} \right|,$$

which gives (2.4) and hence follows that

$$\frac{z(D^n f(z))'}{D^n f(z)} = -\frac{1 + [B + (A-B)(1-\alpha)]w(z)}{1 + Bw(z)} \tag{2.8}$$

$$(-1 \le A < B \le 1, A + B \ge 0, w \in H, 0 \le \alpha < 1, z \in U).$$

That is, $f \in \Sigma_p(n, A, B, \alpha)$.

Theorem 2.2. *A function $f \in \Sigma_{p,1}$ is in $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{F_k a_k}{E_k} \le 1. \tag{2.9}$$

Proof. Let $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$. Then for fixed $z_0(-1 < z_0 < 1, z_0 \neq 0)$, $f(z_0) = \frac{a_{-1}}{z_0} + \sum_{k=1}^{\infty} a_k z_0^k$. Since $f(z_0) = \frac{1}{z_0}$, we have $a_{-1} = 1 - \sum_{k=1}^{\infty} a_k z_0^{k+1}$. Since $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$, $f \in \Sigma_p(n, A, B, \alpha)$ and so from Theorem 2.1 and the relation $a_{-1} = 1 - \sum_{k=1}^{\infty} a_k z_0^{k+1}$, we get (2.9).

Conversely, let $f \in \Sigma_{p,1}$ and let (2.9) be satisfied. Since $f(z_0) = \frac{1}{z_0}$, we get $\sum_{k=1}^{\infty} a_k z_0^{k+1} = 1 - a_{-1}$. Substituting for $(1 - a_{-1})$ in (2.9), we get (2.1). By Theorem 2.1, we have $f \in \Sigma_p(n, A, B, \alpha)$ and hence $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$.

Corollary 2.3. *If $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$, then*

$$a_k \le \frac{E_k}{F_k} \quad (k \in N_0 - \{0\}), \tag{2.10}$$

with equality for the function

$$f(z) = \frac{D_k + E_k z^{k+1}}{z F_k} \quad (k \in N_0 - \{0\}). \tag{2.11}$$

Theorem 2.4. *A function $f \in \Sigma_{p,2}$ is in $\Sigma_{p,2}(z_0, n, A, B, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{(D_k - kE_k z_0^{k+1})}{E_k} a_k \le 1. \tag{2.12}$$

Proof. Suppose $f \in \Sigma_{p,2}(z_0, n, A, B, \alpha)$. Then, for fixed $z_0(-1 < z_0 < 1, z_0 \neq 0)$, $f'(z_0) = -\frac{a_{-1}}{z_0^2} + \sum_{k=1}^{\infty} k a_k z_0^{k-1}$. Since $f'(z_0) = -\frac{1}{z_0^2}$, we have $a_{-1} = 1 + \sum_{k=1}^{\infty} k a_k z_0^{k+1}$. Since $f \in \Sigma_{p,2}(z_0, n, A, B, \alpha)$, $f \in \Sigma_p(n, A, B, \alpha)$ and so Theorem 2.1 holds for f. Hence, substituting $a_{-1} = 1 + \sum_{k=1}^{\infty} k a_k z_0^{k+1}$ in (2.1), we get (2.12).

Conversely, let $f \in \Sigma_{p,2}$ and let (2.12) be satisfied. Since $f'(z_0) = -\frac{1}{z_0^2}$, we have $\sum_{k=1}^{\infty} k a_k z_0^{k+1} = a_{-1} - 1$. Substituting the value of $\sum_{k=1}^{\infty} k a_k z_0^{k+1}$ in (2.12), we get (2.1). From Theorem 2.1, $f \in \Sigma_{p,2}(z_0, n, A, B, \alpha)$.

3. Closure Theorems

Theorem 3.1. *The class $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ is closed under convex linear combination.*

Proof. Let the functions $f_i(z) = a_{-1,i}\frac{1}{z} + \sum_{k=1}^{\infty} a_{k,i}z^k$ $(a_{k,i} \geq 0, a_{-1,i} > 0)$ be in the class $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ for $i = 1, 2, \cdots, m$. We have to show that if the function h is defined by $h(z) = \sum_{i=1}^{m} b_i f_i(z)$ $(b_i \geq 0)$, where $\sum_{i=1}^{m} b_i = 1$, then h also belongs to the class $\Sigma_{p,1}(z_0, n, A, B, \alpha)$. From the definition of $h(z)$, we have

$$h(z) = \frac{d_{-1}}{z} + \sum_{k=1}^{\infty} d_k z^k, \tag{3.1}$$

where $d_k = \sum_{i=1}^{m} b_i a_{k,i}$. Since $f_i(z)$ are in $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ for $i = 1, 2, \cdots, m$, we have from Theorem 2.2,

$$\sum_{k=1}^{\infty} \frac{F_k a_{k,i}}{E_k} \leq 1, \; i = 1, 2, \cdots, m. \tag{3.2}$$

Therefore we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{F_k}{E_k}\left(\sum_{i=1}^{m} b_i a_{k,i}\right) &= \sum_{i=1}^{m} b_i \left(\sum_{k=1}^{\infty} \frac{F_k}{E_k} a_{k,i}\right) \\ &\leq \sum_{i=1}^{m} b_i = 1. \end{aligned} \tag{3.3}$$

This show that the function h belongs to the class $\Sigma_{p,1}(z_0, n, A, B, \alpha)$ and the theorem is proved.

By the similar method of Theorem 3.1, we can prove the following.

Theorem 3.2. *The class $\Sigma_{p,2}(z_0, n, A, B, \alpha)$ is closed under convex linear combination.*

Theorem 3.3. *Let $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{D_k + E_k z^{k+1}}{zF_k} \; (k = 1, 2, \cdots). \tag{3.4}$$

Then $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$ if and only if it can be expressed in the form $f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z)$, where each $\mu_k \geq 0$ and $\sum_{k=0}^{\infty} \mu_k = 1$.

Proof. Suppose $f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z)$, where $\mu_k \geq 0$ and $\sum_{k=0}^{\infty} \mu_k = 1$. Then

$$f(z) = \mu_0 f_0(z) + \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{3.5}$$

$$
\begin{aligned}
&= \mu_0 \frac{1}{z} + \sum_{k=1}^{\infty} \mu_k \frac{D_k + E_k z^{k+1}}{zF_k} \\
&= \left(\mu_0 + \sum_{k=1}^{\infty} \frac{D_k}{F_k} \mu_k \right) \frac{1}{z} + \sum_{k=1}^{\infty} \frac{E_k}{F_k} \mu_k z^k.
\end{aligned}
$$

Now

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{F_k}{E_k} a_k &= \sum_{k=1}^{\infty} \frac{F_k}{E_k} \frac{E_k}{F_k} \mu_k = \sum_{k=1}^{\infty} \mu_k \\
&= 1 - \mu_0 \leq 1.
\end{aligned} \tag{3.6}
$$

Also by definition, we have $f_k(z_0) = \frac{1}{z_0}$. Therefore

$$
f(z_0) = \sum_{k=0}^{\infty} \mu_k f_k(z_0) = \sum_{k=0}^{\infty} \mu_k \frac{1}{z_0} = \frac{1}{z_0}. \tag{3.7}
$$

This implies $f \in \Sigma_{p,1}$. By Theorem 2.2, $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$.

Conversely, let $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$. Then $a_{-1} = 1 - \sum_{k=1}^{\infty} a_k z_0^{k+1}$. Define

$$
\mu_k = \frac{F_k}{E_k} a_k, \ k \geq 1 \text{ and } \mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k. \tag{3.8}
$$

From Theorem 2.2, we have $\sum_{k=1}^{\infty} \mu_k \leq 1$ and so $\mu_0 \geq 0$.
Now

$$
\begin{aligned}
f(z) &= \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_k z^k = \mu_0 \frac{1}{z} + \sum_{k=1}^{\infty} \mu_k \frac{1}{z} [1 + \frac{z^{k+1} - z_0^{k+1}}{\mu_k} a_k] \\
&= \mu_0 \frac{1}{z} + \sum_{k=1}^{\infty} \mu_k \frac{D_k + E_k z^{k+1}}{zF_k} \\
&= \mu_0 f_0(z) + \sum_{k=1}^{\infty} \mu_k f_k(z) = \sum_{k=0}^{\infty} \mu_k f_k(z).
\end{aligned} \tag{3.9}
$$

This completes the proof of the theorem.

In a similar manner, we can prove the following theorem.

Theorem 3.4. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$
f_k(z) = \frac{D_k + E_k z^{k+1}}{z(D_k - kE_k z_0^{k+1})} \ (k = 1, 2, \cdots). \tag{3.10}
$$

Then $f \in \Sigma_{p,2}(z_0, n, A, B, \alpha)$ *if and only if it can be expressed in the form* $f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z)$, *where each* $\mu_k \geq 0$ *and* $\sum_{k=0}^{\infty} \mu_k = 1$.

4. The radius of convexity of the classes $\Sigma_{p,i}(z_0, n, A, B, \alpha)(i = 1, 2)$

Theorem 4.1. *Let $f \in \sum_p$. If $f \in \sum_{p,1}(z_0, n, A, B, \alpha)$, then f is meromorphically convex in the disk $|z| < r$, where*

$$r = \inf_k \left[\frac{D_k}{k(k+2)E_k}\right]^{\frac{1}{k+1}}. \tag{4.1}$$

The bound is sharp for the function given by (3.4).

Proof. To prove the theorem, it is sufficient to show that

$$\left|2 + \frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{4.2}$$

for $|z| \leq r$. Then we have

$$\begin{aligned}\left|2 + \frac{zf''(z)}{f'(z)}\right| &= \left|\frac{\sum_{k=1}^{\infty} k(k+1)a_k z^{k-1}}{-\frac{a_{-1}}{z^2} + \sum_{k=1}^{\infty} ka_k z^{k-1}}\right| \\ &\leq \frac{\sum_{k=1}^{\infty} k(k+1)a_k |z|^{k+1}}{|a_{-1} - \sum_{k=1}^{\infty} ka_k z^{k+1}|}.\end{aligned} \tag{4.3}$$

Consider the values of z for which

$$|z| \leq \inf_k \left[\frac{D_k}{k(k+2)E_k}\right]^{\frac{1}{k+1}}, \tag{4.4}$$

that is,

$$|z|^{k+1} \leq \frac{D_k}{k(k+2)E_k} \tag{4.5}$$

holds. Then

$$\sum_{k=1}^{\infty} ka_k |z|^{k+1} \leq \sum_{k=1}^{\infty} \frac{D_k}{(k+2)E_k} a_k. \tag{4.6}$$

Now

$$\sum_{k=1}^{\infty} ka_k |z|^{k+1} < a_{-1}, \tag{4.7}$$

provided

$$\sum_{k=1}^{\infty} \frac{D_k a_k}{(k+2)E_k} < a_{-1}. \tag{4.8}$$

Now if $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$, then

$$\sum_{k=1}^{\infty} \frac{F_k a_k}{E_k} \leq 1. \tag{4.9}$$

or

$$\sum_{k=1}^{\infty} \frac{D_k}{E_k} a_k < 1 - \sum_{k=1}^{\infty} a_k z_0^{k+1} = a_{-1} \tag{4.10}$$

and since

$$\sum_{k=1}^{\infty} \frac{D_k a_k}{(k+2)E_k} < \sum_{k=1}^{\infty} \frac{D_k a_k}{E_k}, \tag{4.11}$$

(4.8) holds. Therefore we can rewrite the denominator of the right hand side of inequality (4.3) for the considered values of z, using the fact that

$$a_{-1} > \sum_{k=1}^{\infty} k a_k |z|^{k+1}. \tag{4.12}$$

Thus

$$\left| 2 + \frac{z f''(z)}{f'(z)} \right| \leq \frac{\sum_{k=1}^{\infty} k(k+1) a_k |z|^{k+1}}{a_{-1} - \sum_{k=1}^{\infty} k a_k |z|^{k+1}} \leq 1, \tag{4.13}$$

if

$$\sum_{k=1}^{\infty} k(k+2) a_k |z|^{k+1} \leq a_{-1}. \tag{4.14}$$

If $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$, (4.14) is equivalent to

$$\sum_{k=1}^{\infty} [k(k+2)|z|^{k+1} + z_0^{k+1}] a_k \leq 1. \tag{4.15}$$

By Theorem 2.2, $f \in \Sigma_{p,1}(z_0, n, A, B, \alpha)$ if and only if

$$\sum_{k=1}^{\infty} \left(\frac{D_k}{E_k} + z_0^{k+1} \right) a_k \leq 1. \tag{4.16}$$

Hence the inequality (4.15) is true if

$$k(k+2)|z|^{k+1} + z_0^{k+1} \leq \frac{D_k}{E_k} + z_0^{k+1} \tag{4.17}$$

for all k, that is, if

$$|z| \leq \left[\frac{D_k}{k(k+2)E_k} \right]^{\frac{1}{k+1}} \tag{4.18}$$

for all k. This result is sharp for the extremal function

$$f_k(z) = \frac{D_k + E_k z^{k+1}}{z F_k} \qquad (k \in N_0 - \{0\}). \tag{4.19}$$

Similarly, we can prove the following theorem.

Theorem 4.2. *Let $f \in \Sigma_p$. If $f \in \Sigma_{p,2}(z_0, n, A, B, \alpha)$, then f is meromorphically convex in the disk $|z| < r$, where*

$$r = \inf_k \left[\frac{D_k}{k(k+2)E_k}\right]^{\frac{1}{k+1}}. \tag{4.20}$$

The bound is sharp for the function given by (3.10).

5. New criteria for $\Sigma_p(n, A, B, \alpha)$

In order to prove our new characterization theorems, we shall need the followinng lemma due to Mogra, Reddy and Juneja [1].

Lemma 5.1. *A function $f(z)$ of the form (1.4) is meromorphically starlike of order β if and only if*

$$\sum_{k=1}^{\infty}(k+\beta)a_k \le (1-\beta)a_{-1} \ (0 \le \beta < 1). \tag{5.1}$$

Theorem 5.2. *A function $f(z)$ of the form (1.4) is in the class $\Sigma_p(n, A, B, \alpha)$ if and only if $(f * g)(z)$ is meromorphically starlike of order β for the function*

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\frac{(1-\beta)D_k}{(k+\beta)E_k}z^k \ (0 \le \beta < 1). \tag{5.2}$$

Proof. In view of Lemma 5.1, a function $f(z)$ of the form (1.4) is starlike of order β if and only if

$$\sum_{k=1}^{\infty}\left(\frac{k+\beta}{1-\beta}\right)a_k \le a_{-1}. \tag{5.3}$$

Thus, using Theorem 2.1, we have

$$f(z) \in \Sigma_p(n, A, B, \alpha) \Leftrightarrow \sum_{k=1}^{\infty}\frac{D_k a_k}{E_k} \le a_{-1} \tag{5.4}$$

$$\Leftrightarrow \sum_{k=1}^{\infty}\left(\frac{k+\beta}{1-\beta}\right)\left(\frac{(1-\beta)D_k}{(k+\beta)E_k}\right)a_k \le a_{-1}$$

$\Leftrightarrow$ $(f * g)(z)$ is meromorphically starlike of order β.

This completes the proof of Theorem 5.2.

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References

[1] M.L. Mogra, T.R. Reddy and O.P. Juneja, "Meromorphic univalent functions with positive coefficients, "*Bull. Austral. Math. Soc.*, 32(1985), 161-176.

[2] N.E. Cho, "On certain class of meromorphic functions with positive coefficients," *Math. Japonica*, 34(1989), 901-907.

[3] B.A. Uralegaddi and M.D. Ganigi, "A new criterion for meromorphic convex functions," *Tamkang J. Math.*, 19(1988), 43-48.

[4] H. Silverman, "Extreme points of univalent with two fixed points", *Trans. Amer. Math. Soc.*, 219(1976), 387-395.

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