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ON EXISTENCE OF POSITIVE SOLUTIONS OF NEUTRAL DIFFERENCE EQUATIONS*

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Abstract. Consider the neutral difference equation

$$\Delta(x_n - cx_{n-m}) + p_n x_{n-k} = 0, \quad n \geq N \tag{$*$}$$

where c and p_n are real numbers, k and N are nonnegative integers, and m is positive integer. We show that if

$$\sum_{n=N}^{\infty} |p_n| < \infty \tag{$**$}$$

then Eq.$(*)$ has a positive solution when $c \neq 1$. However, an interesting example is also given which shows that $(**)$ does not imply that $(*)$ has a positive solution when $c = 1$.

1. Introduction

For the last few years the oscillation and nonoscillation of solutions of delay difference equations are being extensively investigated [1-3,5-8], for a recent survey, we refer to [4]. In particular, the oscillation of solutions of the neutral difference equation

$$\Delta(x_n - cx_{n-m}) + p_n x_{n-k} = 0, \quad n \geq N \tag{1}$$

have been intestigated in [9-10], where c and p_n are real numbers, k and N are nonnegative integers, and m is positive integer. Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$. However, the results for the existence of positive solutions of Eq.(1) are relatively scarce in the literature, we refer to [10,11], see also Cyori and Ladas's book [4].

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Our aim in this paper is to study the existence of positive solutions of Eq.(1). In section 2 we show that

$$\sum_{n=N}^{\infty} |p_n| < \infty \tag{2}$$

implies that Eq.(1) has a positive solution when $c \neq 1$. In section 3, an interesting example is given to show that it is possible that Eq.(1) has no positive solutions under the hypothesis (2) when $c = 1$.

Let $\rho = \max\{m, k\}$, by a solution of (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq -\rho$ and which satisfies Eq.(1) for $n \geq N$. Clearly, if

$$x_n = A_n \text{ for } n = -\rho, \ldots, -1, 0 \tag{3}$$

are given, then Eq.(1) has a unique solution satisfying the initial conditions (3). We assume throughout that p_n cannot be eventualy identically zero. A nontrivial solution $\{x_n\}$ of (1) is said to be oscillatory if for every $N_0 \geq N$ there exists a $n \geq N_0$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory.

2. Positive solutions of Eq.(1)

In this section we study the existence of positive solutions of Eq.(1) with $c \neq 1$. The main result in this section is the following theorem.

Theorem 1. *Assume that* (2) *holds with* $c \neq 1$, *then Eq.*(1) *has a positive solution.*

Proof. The proof of this theorem is rather too long and will be divided into five claims.

Claim 1. Show Theorem 1 for the case $0 \leq c < 1$.

Indeed, choose a positive integer $N_0 \geq N$ sufficiently large such that $N_0 - \rho \geq N$ and

$$\sum_{n=N_0}^{\infty} |p_n| \leq \frac{1-c}{4}$$

Consider the Banach Space l_∞^N of all real sequences $x = \{x_n\}$ where $n \geq N$ with sup norm $\|x\| = \sup_{n \geq N} |x_n|$. We define a subset S in l_∞^N as

$$S = \{x \in l_\infty^N : 2(1-c)/3 \leq x_n \leq 4/3, n \geq N\}$$

Then S is a bounded, closed and convex subset of l_∞^N. Now we define an operator $T : S \to l_\infty^N$. For $x \in S$,

$$Tx_n = \begin{cases} 1 - c + cx_{n-m} + \sum_{i=n}^{\infty} p_i x_{i-k}, & n \geq N_0, \\ Tx_{N_0} & N \leq n \leq N_0. \end{cases}$$

Clearly, T is continuous. For every $x = \{x_n\} \in S, n \geq N_0$, we have

$$\begin{aligned} Tx_n \leq& 1 - c + \frac{4}{3}c + \frac{4}{3}\sum_{i=n}^{\infty}|p_i| \\ \leq& 1 - c + \frac{4}{3}c + \frac{4}{3}\cdot\frac{1-c}{4} = \frac{4}{3} \end{aligned}$$

and

$$Tx_n \geq 1 - c + \frac{4}{3}\cdot\frac{c-1}{4} = \frac{2(1-c)}{3}$$

Hence, $2(1-c)/3 \leq Tx_n \leq 4/3$ for $n \geq N$, and so $TS \subset S$.

Now we will show that T is a contraction mapping on S. In fact, for any $x, y \in S$ and $n \geq N_0$, we have

$$\begin{aligned} |Tx_n - Ty_n| &\leq c|x_{n-m} - y_{n-m}| + \sum_{i=n}^{\infty}|p_i|\cdot|x_{i-k} - y_{i-k}| \\ &\leq (c + \frac{1-c}{4})\|x - y\| = \frac{1+3c}{4}\|x - y\|. \end{aligned}$$

It follows that

$$\|Tx - Ty\| \leq \frac{1+3c}{4}\|x - y\|.$$

Since $0 < (1+3c)/4 < 1$, we see that T is a contraction on S. Therefore, by the Banach contraction princile, T has a fixed point $x \in S$, i.e., $Tx = x$. It is easy to see that $x = \{x_n\}$ is a positive solution of Eq.(1) and so the proof of Claim 1 is complete.

Claim 2. Theorem 1 holds for the case $c > 1$. Let $N_0 \geq N$ be such that $N_0 + m - k \geq N$ and

$$\sum_{n=N_0+m}^{\infty}|p_n| \leq -\frac{1-c}{4}$$

Consider the Banach Space l_∞^N as in the proof of Claim 1. Set

$$S = \{x \in l_\infty^N : c/2 \leq x_n \leq 2c \text{ for } n \geq N\}$$

Then S is a bounded, closed and convex subset of l_∞^N. Define a mapping $T : S \to l_\infty^N$ as following

$$Tx_n = \begin{cases} c - 1 + \frac{1}{c}x_{n+m} - \frac{1}{c}\cdot\sum_{i=n+m}^{\infty}p_i x_{i-k}, & n \geq N_0, \\ Tx_{N_0}, & N \leq n \leq N_0. \end{cases}$$

Clearly, T is continuous. For every $x = \{x_n\} \in S$ and $n \geq N_0$, we have

$$\begin{aligned} Tx_n \leq& c - 1 + \frac{2c}{c} + \frac{1}{c}2c\sum_{i=n+m}^{\infty}|p_i| \\ \leq& c - 1 + 2 + 2\frac{c-1}{4} = \frac{3c+1}{2} < 2c \end{aligned}$$

and

$$Tx_n \geq c - 1 + \frac{1}{c} \cdot \frac{c}{2} - \frac{1}{c} \cdot 2c \cdot \frac{c-1}{4} = \frac{c}{2}, \text{ for } n \geq N_0$$

Hence, $c/2 \leq Tx_n \leq 2c$ for $n \geq N$, and so $TS \subset S$. Now by a proof similar to the proof of Claim 1, we see that, for any $x, y \in S$,

$$\|Tx - Ty\| \leq \frac{3+c}{4c} \cdot \|x - y\|$$

Since $0 < (3+c)/4c < 1$, it follows that T is a contraction on S. Therefore, by the Banach Contraction Princile, T has a fixed point $x = \{x_n\} \in S$. It is easy to see that this x is a positive solution of Eq.(1) and the proof of Claim 2 is complete.

Claim 3. Prove Theorem 1 for the case $-1 < c < 0$.
Let $N_0 \geq N$ be such that $N_0 - \rho \geq N$ and

$$\sum_{n=N_0}^{\infty} |p_n| \leq \frac{1+c}{4}$$

Let l_∞^N be defined as in the proof of Claim 1. Clearly, the set

$$S = \{x \in l_\infty^N : 2(1+c) \leq x_n \leq 4 \text{ for } n \geq N\}$$

is a bounded, closed and convex subset of l_∞^N. Define $T : S \to l_\infty^N$

$$Tx_n = \begin{cases} 3 - c + cx_{n-m} + \sum_{i=n}^{\infty} p_i x_{i-k}, & n \geq N_0 \\ Tx_{N_0}, & N \leq n \leq N_0 \end{cases}$$

Clearly, T is continuous. It is easy to see that T maps S into itself, and for any $x, y \in S$,

$$\|Tx - Ty\| \leq \frac{1-3c}{4} \cdot \|x - y\|$$

As $0 < (1-3c)/4 < 1$, the Banach Contraction Priciple can be applied to obtain a fixed point $x = \{x_n\}$ of T. It is easy to see that this $\{x_n\}$ is a positive solution of Eq.(1). This completes the proof of Claim 3.

Claim 4. Theorem 1 holds for the case $c = -1$.
Indeed, let $N_0 \geq N$ be such that $N_0 + m - k \geq N$ and

$$\sum_{n=N_0+m}^{\infty} |p_n| \leq \frac{1}{4}$$

Let l_∞^N be defined as in the proof of Claim 1.Then

$$S = \{x \in l_\infty^N : 2 \leq x_n \leq 4 \text{ for } n \geq N\}$$

is a bounded, closed and convex subset of l_∞^N. Now we deifne a mapping as following

$$Tx_n = \begin{cases} 3 + \sum_{j=1}^{\infty} \cdot \sum_{i=n+(2j-1)m}^{n+2mj-1} p_i x_{i-k}, & n \geq N_0 \\ Tx_{N_0}, & N \leq n \leq N_0 \end{cases}$$

Since, for any $x = \{x_n\} \in S$ and $n \geq N_0$,

$$\begin{aligned} Tx_n \leq & 3 + \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)m}^{n+2mj-1} 4|p_i| \\ \leq & 3 + 4 \cdot \sum_{i=N_0+m}^{\infty} |p_i| \\ \leq & 3 + 4 \cdot \frac{1}{4} = 4, \end{aligned}$$

and

$$\begin{aligned} Tx_n \geq & 3 - 4 \cdot \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)m}^{n+2mj-1} |p_i| \geq 3 - 4 \cdot \sum_{i=N_0+m}^{\infty} |p_i| \\ \geq & 3 - 4 \cdot \frac{1}{4} = 2, \end{aligned}$$

it follows that T maps S into S. It is also not difficult to see that for any $x, y \in S$ we have

$$\|Tx - Ty\| \leq \frac{1}{4} \cdot \|x - y\|$$

Therefore, the Banach Contraction Princile can be applied to obtain a fixd point $x \in S$ of T, that is,

$$x_n = \begin{cases} 3 + \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)m}^{n+2mj-1} p_i x_{i-k}, & n \geq N_0 \\ x_{N_0}, & N \leq n \leq N_0 \end{cases}$$

It follows that

$$\begin{aligned} x_n + x_{n-m} = & 6 + \sum_{j=1}^{\infty} \left[\sum_{i=n-m+(2j-1)m}^{n-m+2mj-1} p_i x_{i-k} + \sum_{i=n+(2j-1)m}^{n+2mj-1} p_i x_{i-k} \right] \\ = & 6 + \sum_{i=n}^{\infty} p_i x_{i-k}, \text{ for } n \geq N_0 + m, \end{aligned}$$

From this we see that $x = \{x_n\}$ is a positive soluton of Eq.(1) on $n \geq N_0 + m$, and so the proof of Claim 4 is complete.

Claim 5. Complete the proof of Theorem 1 when $c < -1$.

Let $N_0 \geq N$ be such that $N_0 + m - k \geq N$ and

$$\sum_{n=N_0+m}^{\infty} |p_n| \leq \frac{-c-1}{4}$$

Let l_∞^N be defined as in the proof of Claim 1 and let

$$S = \{x \in l_\infty^N : -2(c+1) \leq x_n \leq -4c, n \geq N\}$$

Clearly, S is a bounded, closed and convex subset of l_∞^N. Define a mapping T on S as following

$$Tx_n = \begin{cases} -3c + 1 + \frac{1}{c}x_{n+m} + \sum_{i=n+m}^{\infty} p_i x_{i-k}, & n \geq N_0 \\ Tx_{N_0}, & N \leq n \leq N_0 \end{cases}$$

By an argument similar to that in the proof of Claim 2 we can easily show that the all hypotheses of the Banach Contraction Principle are satisfied. Therefore, T has a fixed point $x = \{x_n\} \in S$. It is easy to see that this $\{x_n\}$ is a positive solution of Eq.(1) for $n \geq N_0 + m$ and the proof of Claim 5 is complete.

Combining Claim 1-5, we see that the proof of Theorem 1 is complete.

3. An example

The aim in the section is to show by the folloing example that Theorem 1 does not hold when $c = 1$.

Example 1. Consider the neutral difference equation

$$\Delta(x_n - x_{n-1}) + \frac{1}{nln^2n}x_{n-1} = 0, n \geq 2 \tag{4}$$

Here $m = k = 1, c = 1$, and $p_n = 1/n\ln^2 n$, Since

$$\sum_{n=2}^{\infty} p_n = \sum_{n=2}^{\infty} \frac{1}{n\ln^2 n} < \infty,$$

it follows that (2) holds. Next we will prove that Eq.(4) has no positive solutions. Otherwise, assume that (4) has a positive solution $= \{x_n\}$ satisfying $x_{n-1} > 0, n \geq N_0 \geq 2$, for some $N_0 \geq 2$. Set

$$z_n = x_n - x_{n-1}, \text{ for } n \geq N_0$$

Then by (4) we have

$$\Delta z_n = -\frac{1}{n\ln^2 n}x_{n-1} < 0 \quad \text{for} \quad n \geq N_0 \tag{5}$$

We consider the following two possible cases:

(i) If z_n is eventually negative, then by (5) we see that there exist $\alpha > 0$ and positive integer $N_1 \geq N_0$, such that

$$z_n \leq -\alpha \text{ for } n \geq N_1$$

This is

$$x_n \leq -\alpha + x_{n-1}, \text{ for } n \geq N_1$$

By using the Induction Principle, we get

$$x_{n+N_1} \leq -n\alpha + x_{N_1} \to -\infty \text{ as } n \to \infty$$

which contradicts the positivity of $\{x_n\}$.

(ii) If z_n is eventually positive, then there exists positive integer $N_2 \geq N_1$ such that $z_n > 0$ for $n \geq N_2$. Now we let

$$\lim_{n\to\infty} z_n = \beta \in [0, \infty)$$

and sum (5) from $n \geq N_2$ to ∞, we have

$$\beta - z_n + \sum_{k=n}^{\infty} \frac{1}{k \ln^2 k} x_{k-1} = 0 \tag{6}$$

this implies that

$$\sum_{n=N_2}^{\infty} \frac{1}{n \ln^2 n} x_{n-1} < \infty$$

On the other hand, since $x_n - x_{n-1} > 0$, for $n \geq N_2$, it follows that there exists a positive constant M such that $x_n \geq M$ for $n \geq N_2$, Substituting this into (6) we find that

$$\begin{aligned} z_n \geq& \beta + M \sum_{k=n}^{\infty} \frac{1}{k \ln^2 k} \geq \beta + M \int_n^{\infty} \frac{1}{t \ln^2 t} dt \\ =& \beta + \frac{M}{\ln n,}, \text{ for } n \geq N_2 + 1, \end{aligned}$$

that is,

$$x_n \geq x_{n-1} + \frac{M}{\ln n}, \text{ for } n \geq N_2 + 1$$

It follows that

$$\begin{aligned} x_n \geq& M \left[\frac{1}{\ln n} + \frac{1}{\ln(n-1)} + \cdots + \frac{1}{\ln(n+1-(n-N_2))} \right] + x_{N_2} \\ \geq& M \frac{n - N_2}{\ln n}, \text{ for } n \geq N_2 + 1 \end{aligned}$$

Hence, we find that

$$\frac{1}{n\ln^2 n}x_{n-1} \geq M\frac{n-N_2-1}{n\ln^2 n\ln(n-1)} \text{ for } n \geq N_2+2 \tag{7}$$

As $\sum_{n=N_2}^{\infty}\frac{n-N_2-1}{n\ln^2 n\ln(n-1)} = \infty$, it follows from (7) that

$$\sum_{n=N_2}^{\infty}\frac{1}{n\ln^2 n}x_{n-1} = \infty$$

This contradiction shows that the case (ii) is also impossible. The proof which Eq.(4) has no positive solutions is complete.

The following example 2 shows that it is also possible that Eq.(1) has a positive solution when (2) holds and $c = 1$.

Example 2. Consider the neutral difference equation

$$\Delta(x_n - x_{n-1}) + p_n x_{n-1} = 0, n \geq 1$$

Here $p_n = \frac{1}{n(n+1)}(\sum_{i=1}^{n-1}\frac{1}{i})^{-1}$ It is obvious that (2) holds, and this equation has a positive solution $x_n = \sum_{i=1}^{n} 1/i, n = 1, 2, \cdots$

Remark. Combining the Theorem 1 and Example 1, we know that

$$\sum_{n=N}^{\infty}|p_n| = \infty$$

is an necessary condition for the oscillation of all solutions of Eq.(1) when $c \neq 1$.

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