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ON COSYMPLECTIC CAUCHY-RIEMANN SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER MANIFOLDS

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Abstract. We study some properties of the cosymplectic Cauchy-Riemann submanifolds in a locally conformal Kaehler manifold.

Introduction

The geometry of the Cauchy-Riemann (C.R.) submanifolds of a locally conformal Kaehler (l.c.K.) manifold has been studied in the last ten years, ([4], [5], [6], [10], [12], [13], [18], [19]).

The concept of normal C.R. submanifold was introduced by A. Bejancu ([1]) in analogy with the theory of the normal almost contact structures, ([3], [7]).

In [1] a theory for the normal C.R. submanifolds in a Kaehler manifold is developed. In particular, a C.R. hypersurface of a Kaehler manifold is a normal contact hypersurface, ([14]).

Some properties of the normal C.R. submanifolds of l.c.K. manifolds have been studied in former papers, ([18], [19]).

In this paper, we study the cosymplectic C.R. submanifolds in a l.c.K. manifold.

1. Preliminaries

Let (M^{2n}, g_0, J) be a Hermitian manifold of complex dimension n, with Kaehler 2-form Ω_0, i.e. $\Omega_0(X,Y) = g_0(X, JY), X, Y \in TM^{2n}$.

Then (M^{2n}, g_0, J) is a *locally conformal Kaehler (l.c.K.) manifold* if there exists a closed 1-form ω_0 on M^{2n} such that

$$d\Omega_0 = \omega_0 \wedge \Omega_0. \qquad (1.1)$$

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The 1-form ω_0 is called the *Lee form*, the *Lee vector field* is the vector field B_0 such that $g_0(B_0, X) = \omega_0(X), X \in TM^{2n}$.

If $\bar{\nabla}$ denotes the Riemannian connection of (M^{2n}, g_0), then one has:

$$(\overline{\nabla}_X J)Y = \frac{1}{2}\{\theta_0(Y)X - \omega_0(Y)JX - \Omega_0(X,Y)B_0 - g(X,Y)A_0\} \quad X, Y \in TM^{2n} \quad (1.2)$$

where $\theta_0 = \omega_0 \circ J$ is the *anti-Lee* 1-*form* and $A_0 = -JB_0$ is the *anti-Lee vector field*. We use the notation and the properties stated in [16], [17].

A submanifold M^m of M^{2n} is called a *Cauchy-Riemann* $(C.R.)$ *submanifold* of M^{2n} if the tangent bundle TM^m is expresed as a direct sum of two distributions O and $O^\perp$, such that O is holomorphic (i.e. $J_x(O_x) = O_x, x \in M^m$) and $O^\perp$ is totally real (i.e. $J_x(O_x^\perp) \subset (T_xM^m)^\perp, x \in M^m$).

Let p be the complex dimension of the holomorphic distribution O and let q be the real dimension of the totally real distribution $O^\perp$.

If $q = 0, M^m$ is called *holomorphic submanifold*; if $p = 0, M^m$ is called *totally real submanifold*.

Let $\tan_x$ and nor_x be the projections naturally associated with the direct sum decomposition $T_xM^{2n} = T_xM^m \oplus (T_xM^m)^\perp, x \in M^m$.

We put $PX = \tan(JX), FX = \mathrm{nor}(JX), t\xi = \tan(J\xi)$ and $f\xi = \mathrm{nor}(J\xi)$ for any $X \in TM^m, \xi \in (TM^m)^\perp$.

Then, for any $X \in TM^m$ one has $PX \in O$.

Moreover, the following identities hold: $P^2 = -I - tF, f^2 = -I - Ft, FP = 0, fF = 0, tF = 0, Pt = 0, P^3 + P = 0, f^3 + f = 0$, ([11]).

The Gauss and Weingarten formulas are still valid, that is:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi \quad (1.3)$$

for any $X, Y \in TM^m, \xi \in (TM^m)^\perp$.

Here ∇, h, A_ξ and $\nabla^\perp$ stand, respectively, for *the induced connection, the second foundamental form, the Weingarten operator* (associated with $\xi \in (TM^m)^\perp$) and *the normal connection* in $(TM^m)^\perp$.

The forms θ, ω and Ω are naturally induced on the submanifold M^m by θ_0, ω_0 and Ω_0 respectively. One has:

$$\theta = \omega \circ P + \omega_0 \circ F, \quad \Omega(X,Y) = g(X, PY), \quad X, Y \in TM^m \quad (1.4)$$

As a consequence of (1.2) and (1.3) one has:

$$(\nabla_X P)Y = A_{FY}X + \mathrm{th}(X,Y) + \frac{1}{2}\{\theta(X)Y - \omega(Y)PX - \Omega(X,Y)B - g(X,Y)A\} \quad (1.5)$$

$$(\nabla_X F)Y = fh(X,Y) - h(X,PY) - \frac{1}{2}\{\omega(Y)FX + \Omega(X,Y)B^\perp + g(X,Y)A^\perp\} \quad (1.6)$$

for any $X, Y \in TM^m$, where $A = \tan(A_0)$, $B = \tan(B_0)$, $A^\perp = \text{nor}(A_0)$ and $B^\perp = \text{nor}(B_0)$. We put:

$$S(X,Y) = [P,P](X,Y) - 2t(dF)(X,Y), \quad X,Y \in TM^m. \tag{1.7}$$

Here $[P,P]$ is the Nijenhuis torsion of P and dF is the differential of the vector valued 1-form F, which can be expressed as follows:

$$2(dF)(X,Y) = \nabla^\perp_X(FY) - \nabla^\perp_Y(FX) - F[X,Y], \quad X,Y \in TM^m. \tag{1.8}$$

A C.R. submanifold is called *normal* if $S = 0$, ([1]).

A C.R. submanifold is called *cosymplectic* if it is normal and F and Ω are closed. P is called *parallel* if $\nabla P = 0$. F is called *parallel* if $\nabla F = 0$.

2. Cosymplectic C.R. submanifolds

Lemma 2.1. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. Then, one has:*

$$\begin{aligned} 2g((\nabla_X P)Y, Z) =& 3(d\Omega)(X, PY, PZ) - 3(d\Omega)(X,Y,Z) + g([P,P](Y,Z), PX) \\ &+ 2g_0((dF)(PY,Z), FX) + 2g_0((dF)(PY,X), FZ) \\ &- 2g_0((dF)(PZ,X), FY) - 2g_0((dF)(PZ,Y), FX) \end{aligned} \tag{2.1}$$

for any $X, Y, Z \in TM^m$.

By a easy calculation one has:

$$2(dF)(PY,Z) = \nabla^\perp_{PY}(FZ) - F[PY,Z], \quad Y,Z \in TM^m$$

$$\begin{aligned} 3(d\Omega)(X,Y,Z) =& X(g(Y,PZ)) + Y(g(Z,PX)) + Z(g(X,PY)) - g([X,Y],PZ) \\ &- g([Z,X],PY) - g([Y,Z],PX), \quad X,Y,Z \in TM^m. \end{aligned}$$

Then, for any $X, Y, Z \in TM^m$ it follows that:

$$\begin{aligned} &3(d\Omega)(X,PY,PZ) - 3(d\Omega)(X,Y,Z) + g([P,P](Y,Z),PX) + 2g_0((dF)(PY,Z),FX) \\ &+ 2g_0((dF)(PY,X),FZ) - 2g_0((dF)(PZ,X),FY) - 2g_0((dF)(PZ,Y),FX) \\ =& X(g(PY, -Z - tFZ)) + (PY)(g(Z,X) - g_0(FZ,FX)) - (PZ)(g(PX,PY)) \\ &+ g(P[X,PY],PZ) + g(P[PZ,X],PY) - g([PY,PZ],PX) + X(g(PY,Z)) \\ &- Y(g(Z,PX)) - Z(g(X,PY)) + g([X,Y],PZ) + g([Z,X],PY) + g([Y,Z],PX) \\ &+ g([PY,PZ],PX) - g(P[PY,Z],PX) - g(P[Y,PZ],PX) + g(P^2[Y,Z],PX) \\ &+ 2g_0((dF)(PY,Z),FX) + 2g_0((dF)(PY,X),FZ) - g_0(\nabla^\perp_{PZ}(FX),FY) \end{aligned}$$

$$
\begin{aligned}
&+ g_0(F[PZ,X],FY) - g_0(\nabla^{\perp}_{PZ}(FY),FX) + g_0(F[PZ,Y],FX)\\
=&(PY)(g(Z,X)) - (PY)(g_0(FZ,FX)) - (PZ)(g(X,Y)) + g(P[X,PY],PZ)\\
&+ g([PZ,X],Y) - Y(g(Z,PX)) - Z(g(X,PY)) + g([X,Y],PZ)\\
&+ g([Z,X],PY) - g([PY,Z],X) + g_0(F[PY,Z],FX) - g([Y,PZ],X)\\
&+ 2g_0((dF)(PY,Z),FX) + 2g_0((dF)(PY,X),FZ)\\
=&X(g(PY,Z)) + (PY)(g(Z,X)) - Z(g(X,PY)) + g([X,PY],Z)\\
&+ g([Z,X],PY) - g([PY,Z],X) + X(g(Y,PZ)) + Y(g(PZ,X))\\
&- (PZ)(g(X,Y)) + g([X,Y],PZ) + g([PZ,X],Y) - g([Y,PZ],X)\\
=&2g((\nabla_X P)Y,Z).
\end{aligned}
$$

Theorem 2.1. *Let M^m be a C.R. submanifold of th l.c.K. manifold M^{2n}. If M^m is cosymplectic then P is parallel.*

The statement is a consequence of the lemma 2.1.

Theorem 2.2. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is cosymplectic then M^m is a C.R. product.*

The statement is a consequence of the theorem 2.1 and of the theorem 5.1 in [12].

Remark 2.1. Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If $dF = 0$ and M^m is normal, by (1.7), one has: $[P,P] = 0$.

Proposition 2.1. *Let M^m be a C.R. submanifold of a l.c.K. manifold M^{2n}. Then one has:*
$S(X,Y) = (\nabla_{PX}P)Y - (\nabla_{PY}P)X + P((\nabla_Y P)X - (\nabla_X P)Y) - J((\nabla_X F)Y - (\nabla_Y F)X)$
for any $X, Y \in TM^m$.

For any $X, Y, \in TM^m$, one has:

$$
\begin{aligned}
&(\nabla_{PX}P)Y - (\nabla_{PY}P)X + P((\nabla_Y P)X - (\nabla_X P)Y) - J((\nabla_X F)Y - (\nabla_Y F)X)\\
=&[PX,PY] - P(\nabla_{PX}Y - \nabla_Y PX) - P(\nabla_X(PY) - \nabla_{PY}X) + P^2(\nabla_X Y - \nabla_Y X)\\
&- J(\nabla^{\perp}_X(FY) - \nabla^{\perp}_Y(FX) - F(\nabla_X Y - \nabla_Y X))\\
=&S(X,Y).
\end{aligned}
$$

Corollary 2.1. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If P and F are parallel then M^m is normal.*

Remark 2.2. Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If $[P,P] = 0$, then the distribution O is integrable.
If the distribution O is integrable then $[P,P](X,Y) \in O$ for any $X, Y \in TM^m$.

Corollary 2.2. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If P and F are parallel then the distribution O is integrable.*

Theorem 2.2. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If P and F are parallel then M^m is cosymplectic.*

The corollary 2.1 implies that M^m is normal.

Since F is parallel it follows that $dF = 0$, and the remark 2.1 gives $[P, P] = 0$. By means of the lemma 2.1 one has:

$$(d\Omega)(X, Y, Z) = (d\Omega)(X, PY, PZ), \quad X, Y, Z \in TM^m. \tag{2.2}$$

Putting $X = Y$, one has:

$$(d\Omega)(X, PX, PZ) = 0, \qquad X, Z \in TM^m. \tag{2.3}$$

Replacing $X + Y$ with X in (2.3), one has:

$$(d\Omega)(X, PY, PZ) = -(d\Omega)(Y, PX, PZ), \quad X, Y, Z \in TM^m; \tag{2.4}$$

$$(d\Omega)(X, Y, Z) = 0, \; X, Z \in TM^m, \qquad Y \in O^\perp. \tag{2.5}$$

Applying (2.2) and (2.5) for any $X, Y, Z \in TM^m$ one has:

$$\begin{aligned}(d\Omega)(X, Y, Z) =& (d\Omega)(X, PY, PZ) = -(d\Omega)(Y, PX, PZ) = -(d\Omega)(PX, PZ, Y)\\ =& -(d\Omega)(PX, P^2Z, PY) = (d\Omega)(PX, Z, PY) = -(d\Omega)(Z, PX, PY)\\ =& -(d\Omega)(X, Y, Z).\end{aligned}$$

Therefore Ω is closed.

Propositon 2.2. *Let M^m be a C.R. submanifold of the l.c.K. manifold M^{2n}. If M^m is generic, $q > 1$, normal and P is parallel then F is parallel.*

Since P is parallel it follows that $\nabla_X Y \in O$ for any $X \in TM^m, Y \in O$. Therefore one has $(\nabla_X F)Y = 0$ for any $X \in TM^m, Y \in O$.

By (1.7) one has:

$$(\nabla_X F)Y = fh(X, PY) - \frac{1}{2}\{\omega(Y)FX + \Omega(X, Y)B^\perp + g(X, Y)A^\perp\}, \; X, Y \in TM^m.$$

Applying the theorem 2.1 in [6] one obtains $\omega = 0$ on the distribution $O^\perp$ and $A^\perp = 0$.

Since M^m is generic, it follows that $f = 0$.

Therefore one has $(\nabla_X F)Y = 0$ for any $X \in TM^m, Y \in O^\perp$.

Corollary 2.3. *Let M^m be a generic C.R. submanifold, $q > 1$, of the l.c.K. manifold M^{2n}. The following statements are equivalent:*

a) *M^m is cosymplectic;*

b) *P and F are parallel.*

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