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ON THE DENJOY-PERRON-BOCHNER INTEGRAL

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Abstract. The notion of Denjoy integrals of abstract functions was first introduced by *A.* Alexiewicz [1]. His descriptive definitions are based upon a concept of the approximate derivative. In this paper we present another descriptive definition for the Denjoy-Perron integral of abstract functions – via the parametric derivative of Tolstov [8]. Some properties of this integral are examined.

We assume that the reader is familiar with [2]. Let X be a Banach space. The definition of Tolstov ([8] p.387) and the important theorem of Armstrong ([2] p.36, Theorem 2) lead to the following definition.

Definition. *Let $F : [a,b] \to X$, where $[a,b]$ is a finite interval. The function $f : [a,b] \to X$ is called the parametric derivative of F if there exists a differentiable strictly increasing function φ mapping $[\alpha,\beta]$ onto $[a,b]$ such that*

$$\frac{d}{dt}F(\varphi(t)) = \varphi'(t) \cdot f(\varphi(t)) \qquad \textit{for every} \qquad t \in [\alpha,\beta].$$

The function will be called a dpr for F (dpr stands for differentiable parametric representation). We shall write $D(F) = f$.

The properties of the parametric derivative are the same as in [2], namely we have

a. If $F' = f$ then $\varphi(t) = t$ is a dpr for F.
b. If F has a parametric derivative, then it is continuous.
c. If $D(F) = f$, and F has dpr φ, and $\varphi'(t) \neq 0$, then F has an ordinary derivative at the point $x = \varphi(t)$, $F'(t) = f(x)$.
d. If $D(F) = f$ on $[a,b]$, then a.e. on $[a,b]$ F has an ordinary derivative $F' = f$ (cit. [2] pp.31, 32).

The proofs of the above properties run as in [2]. The properties 1, 2, 3 ([2] p.31) remain also true, i.e. we have

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1. $D(kF) = kD(F)$, k is a constant,
2. $D(F+G) = D(F) + D(G)$,
3. $D(F) = 0$ implies F is a constant.

To prove 2 define $\Theta = ((\varphi^{-1} + \psi^{-1}/2)^{-1}$, where φ, ψ are dpr for F, G, respectively. For every h belonging to the set $\{\varphi^{-1}, \psi^{-1}, \Theta^{-1}\}$ h is a strictly increasing bijection $[a,b]$ onto $[\alpha,\beta]$ and $0 < h' \leq \infty$, so $0 \leq (h^{-1})' < \infty$ and h^{-1} is a strictly increasing bijection $[\alpha,\beta]$ onto $[a,b]$. Thus Θ satisfies the assumptions of our definition. Obviously we have $0 < 2(\Theta^{-1})' = (\varphi^{-1})' + (\psi^{-1})' \leq \infty$, hence $(\Theta^{-1})'(x) = \infty$ if and only if $(\varphi^{-1})'(x) = \infty$ or/and $(\psi^{-1})'(x) = \infty$. Therefore, if $x = \Theta(t) = \varphi(s) = \psi(r)$ then we obtain

$$\Theta'(t) = 0 \quad \text{if and only if} \quad \varphi'(s) = 0 \quad \text{or/and} \quad \psi'(r) = 0.$$

The above, the Definition and property c imply that Θ is a dpr for both F and G, because $(F \circ \Theta)'(t)$ and $(G \circ \Theta)'(t)$ exist. To prove the last statement we consider for example the case $\Theta'(t) = 0$, $\varphi'(s) = 0$, $\psi'(r) \neq 0$. Since $x = \Theta(t) = \varphi(s) = \psi(r)$ and $\Theta(t+h) = \Theta(t) + o(h)$, $\varphi(s+h_1) = \varphi(s) + o_1(h_1)$, $\psi(r+h_2) = \psi(r) + \psi'(r)h_2 + o_2(h_2)$, we obtain for $\Theta(t+h) = \varphi(s+h_1) = \psi(r+h_2)$ the following equalities $o(h) = o_1(h_1) = o_2(h_2) + \psi'(r)h_2$. According to the definition of Θ we get $t = (s+r)/2$, $t+h = (s+h_1+r+h_2)/2$ and the signs of h, h_1, h_2 are the same. Therefore $2h = h_1 + h_2$ and h_1/h is bounded as $h \to 0$ (moreover, $h \to 0$ if and only if $h_1 \to 0$). Consequently,

$$\begin{aligned}(F \circ \Theta)'(t) &= \lim_{h\to 0} (F(x + o(h) - F(x))/h = \lim_{h\to 0} \frac{h_1}{h} \cdot \frac{F(x + o_1(h_1)) - F(x)}{h_1} \\ &= \lim_{h\to 0} \frac{h_1}{h} \cdot \frac{F(\varphi(s+h_1)) - F(\varphi(s))}{h_1} = 0,\end{aligned}$$

because of $(F \circ \varphi)'(s) = 0$. This proves the existence of $(F \circ \Theta)'(t)$. The existence of $(G \circ \Theta)'(t)$ is a consequence of the property c.

Finally, Θ is a dpr for both F and G. Thus $((F+G) \circ \Theta)' = (F \circ \Theta + G \circ \Theta)' = (F \circ \Theta)' + (G \circ \Theta)'$, which yields property 2 (cf. [2], p. 32). Properties 1 and 3 are easy to prove (cf. [2], pp. 31, 32).

Let X_i $(i = 1,2,3)$ be a Banach space, $F_i : [a,b] \to X_i (i = 1,2)$, $B : X_1 \times X_2 \to X_3$ is an arbitrary bilinear and continuous map. Then we have also

e. $D(B(F_1, F_2)) = B(F_1, D(F_2)) + B(D(F_1), F_2)$.

The proof is similar to that of [2] p.32 if we use a formula on the derivative of bilinear product from [7], Chapter III, 4.

Let f be a parametric derivative of F on $[a,b]$. We define the Denjoy-Perron-Bochner integral (short: DPB-integral) of f by

$$\int_a^b f = F(b) - F(a)$$

(see [2] p.32 (3) for real-valued functions). The function f above is said to be Denjoy-Perron-Bochner integrable (short: DPB-integrable). This integral is well-defined (see the proof of Theorem 1, [2] p.33). Moreover, theorem 2, 3, 4, 5 from [2] pp. 33, 34 remain also true with the similar proofs (obviously the proof of 4 must be suitable changed). Therefore the following properties hold for DPB integral:

α. if f, g are DPB-integrable in $[a,b]$, k is a constant, then $kf, f+g$ are also DPB-integrable on $[a,b]$ and

$$\int_a^b kf = k\int_a^b f, \quad \int_a^b (f+g) = \int_a^b f + \int_a^b g,$$

β. if f is DPB-integrable in $[a,c]$ and on $[c,b]$, then f is DPB-integrable on $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f,$$

γ. if F_1, F_2 are the same as in the property d, $D(F_1), D(F_2)$ exist and one of summands on the right side of d is DPB-integrable, so is the other, and

$$\int_a^b B(F_1, D(F_2)) = B(F_1(b), F_2(b)) - B(F_1(a), F_2(a)) - \int_a^b B(D(F_1), F_2)),$$

δ. if f is the derivative of F on $[a,b]$, then f is DPB-integrable and

$$\int_a^b f = F(b) - F(a).$$

In the sequel the Bochner integral of f on $[a,b]$ will be denoted by $(B)\int_a^b f$. Using Th. 8 of Zahorski ([10] p.35) and arguing as in [2] p.35 we obtain

ε. if $f = 0$ a.e. on $[a,b]$, then f is DPB-integrable on $[a,b]$ and $\int_a^b f = 0$. Now, we shall prove the following

Theorem A. *A DPB-integrable function f is strongly measurable.*

Proof. Let $f = D(F)$ be defined on $[a,b]$, and $Z = \{x \in (a,b) : F'(x)$ exists and $F'(x) = f(x)\}$. By the property d, the complement of the set Z to the set $[a,b]$ has Lebesgue measure zero. Define $h_n = (b-a)/n, I(n,k) = [a+(k-1)h_n, a+kh_n], \Delta I(n,k) = h_n, \Delta F(n,k) = F(a+kh_n) - F(a+(k-1)h_n), y_n(x) = \sum_{k=1}^{n} \chi_{n,k}(x)$ $\Delta F(n,k)/\Delta I(n,k)$, where $\chi_{n,k}$ is the characteristic function of the interval $I(n,k)$.

Fix $x \in Z$. For every n there exists an interval $I(n,k(t))$ containing t, where t is a member of (a,b). Moreover, the diameter of this interval tends to zero as n tends to infinity. Thus

$$\lim_{n\to\infty} \frac{\Delta F(n,k(x))}{\Delta I(n,k(x))} = F'(x) \qquad \text{(see [5] } p.157).$$

Consequently, the sequence of simple functions y_n tends to $F'(x) = f(x)$ for almost all $x \in [a,b]$ and therefore $f(x)$ is strongly measurable on $[a,b]$ ([9] p. 130).

Theorem B. *If $f : [a,b] \to X$, the dimension of X is finite and f is Bochner integrable on $[a,b]$, then f is DPB-integrable there and the two integrals are equal.*

Proof. Let $e_1, \dots, e_n$ be a basis of X and $f = \sum_{i=1}^{n} f_i e_i$. From this and Th. 3 [2] it follows that

$$(B)\int_a^b f = \sum_{i=1}^{n}\Big((L)\int_a^b f_i\Big)\cdot e_i = \sum_{i=1}^{n}\Big(\int_a^b f_i\Big)e_i = \sum_{i=1}^{n}\int_a^b (f_i e_i) = \int_a^b f$$

by the property 2 and e.

Theorem C. *Let T be a bounded linear operator on a Banach space X into a Banach space Y. If f is an X-valued and DPB-integrable on $[a,b]$ function, then Tf is a Y-valued DPB-integrable function, and*

$$T\Big(\int_a^b f\Big) = \int_a^b T \circ f.$$

Proof. Let F be a parametric primitive of f with dpr φ. Now T is continuous and therefore it is commutative with the derivative. We have $((T \circ F) \circ \varphi)'(t) = (T \circ (F \circ \varphi))'(t) = T((F \circ \varphi)'(t)) = T(\varphi'(t) f(\varphi(t))) = \varphi'(t) \cdot [T \circ f](\varphi(t))$, whence φ is a dpr for $T \circ F$ and $D(T \circ F) = T \circ f$. Thus

$$\int_a^b T \circ f = (T \circ F)(b) - (T \circ F)(a) = T(F(b) - F(a)) = T\Big(\int_a^b f\Big)$$

and the proof is finished.

Theorem D. *If f is Bochner integrable on $[a,b]$ and f is DPB-integrable on $[a,b]$, then the two integrals are equal there.*

Proof. Denote the first integral by u, the second one by v both on $[a,b]$. Let T be a bounded linear functional defined on X.
Define

$$p = (B)\int_a^b T \circ f, \quad q = \int_a^b T \circ f.$$

It follows from Th. C, Corollary 2 of [9] p. 134 and Th. 3 of [2] that

$$T(u - v) = Tu - Tv = p - q = 0.$$

Therefore $u - v$ must be 0 (since T is an arbitrary element of the dual space of X). The proof is finished.

From the descriptive definitions of the Bochner and the Denjoy-Bochner integrals ([1], pp. 101, 102) it follows immediately that if f is Bochner integrable on $[a,b]$, then f is DB-integrable there and the two integrals are equal. But then the DB-integral is a generalization of the general Denjoy integral, whereas our DPB-integral is a generalization of the restricted Denjoy integral. These considerations and Theorems B, D suggest the following theorem.

Theorem E. *If f is Bochner integrable on $[a,b]$, then f is DPB-integrable there and the two integrals are equal.*

Proof. Let $\|\ \|$ be the norm in the space X, $A = (B)\int_a^b \|f(x)\|dx$. If $A = 0$, then $\|f(x)\| = 0$ for almost all x in $I = [a,b]$, so $f = 0$ a.e. on I. Thus the two integrals are equal by the property ε (see also [9] p. 133).

Suppose $A > 0$. Since the function $c(t) = t + (B)\int_a^t \|f(x)\|dx$ is increasing and continuous on I, then for every s from $J = [a, b+A]$ there exists $t \in I$, such that $s = c(t)$. Denoting this t by $d(s)$, we have the increasing bijection $d : J \to I$. Let us consider the function $F(t) = (B)\int_a^t f(x)dx$. Taking $s_1 \leq s_2$ from J, we obtain for $t_i = d(s_i)$, $(i = 1,2)$

$$|d(s_2) - d(s_1)| = t_2 - t_1 \leq t_2 - t_1 + (B)\int_{t_1}^{t_2} \|f(x)\|dx = |s_2 - s_1|,$$

and similarly

$$\|F(d(s_2)) - F(d(s_1))\| = \|(B)\int_{t_1}^{t_2} f(x)dx\| \leq t_2 - t_1 + (B)\int_{t_1}^{t_2} \|f(x)\|dx = |s_2 - s_1|.$$

From this it follows that the funcitons $d, F \circ d$ satisfy the Lipschitz condition on J. Moreover, d (as well as c) is absolutely continuous.

The set $E = \left\{x \in I : F'(x) = f(x)\right\}$ has measure $b - a$, thus the set $Z = I \backslash E$ has measure zero (Th. 2, [9], p. 134). Therefore the set $d^{-1}(Z) = c(Z)$ has measure zero (Th. 1, [5] p. 172) and $(F \circ d)'(s)$ exists for almost all $s \in J$. Indeed, $d'(s)$ exists for almost all $s \in J$ and therefore $(F \circ d)'(s) = F'(d(s)) \cdot d'(s)$ for almost all $s \in J \backslash d^{-1}(Z)$, thus for almost all $s \in J$ (cf. [5], Corollary, p. 173). Repeating the arguments of Bruckner ([4], p. 555, $\text{lines}_{17,21}$) we can find a dpr φ for $F \circ d$ and a dpr ψ for d (both $F \circ d$ and d satisfy the Lipschitz condition). Let $\Theta : K \to J$ be a common dpr for $F \circ d$, d (compare with the proof of property 2). Thus

$$(F \circ d \circ \Theta)'(r) \quad \text{exists for all} \quad r \in K. \tag{1}$$

Denote by C, G the sets

$$\left\{x \in I \backslash E : x = d \circ \Theta(r),\ (d \circ \Theta)'(r) = 0 \qquad \text{for some} \qquad r \in K\right\},$$

$$\left\{x \in I \backslash E : x = d \circ \Theta(r),\ (d \circ \Theta)'(r) \neq 0 \quad \text{for some} \quad r \in K\right\},$$

respectively. The sets E, C, G are disjoint and $E \cup C \cup G = I$, since $(d \circ \Theta)'(r)$ exists for all $r \in K$. Define the function $g(x)$ to be $f(x), 0, ((d \circ \Theta)'(r))^{-1} \cdot (F \circ d \circ \Theta)'(r)$ for $x = d \circ \Theta(r)$ belonging to E, C, G, respectively. From this definition and (1) we obtain

$$(F \circ (d \circ \Theta))'(r) = g(d \circ \Theta(r)) \cdot (d \circ \Theta)'(r) \quad \text{for all} \quad r \in K,$$

and therefore $d \circ \Theta$ is a dpr for $F, D(F) = g$. Since $g - f = 0$ a.e. on I, then $g - f$ is DPB-integrable and its DPB-integral is equal to zero (property ε). Finally,

$$F(b) - F(a) = \int_a^b g = \int_a^b (f - g + g) = \int_a^b f \qquad \text{(property } \alpha\text{), so}$$

$$(B) \int_a^b f(x)dx = F(b) - F(a) = \int_a^b f, \qquad (q.e.d.)$$

The theorem on integration by substitution ([2], p.38) can be reformulated as follows.

Theorem F. *Let $g : [c, d] \to \mathbf{R}$ be Lebesgue integrable and positive on $[c, d]$, $G(t) = a + \int_0^t g$ for $c \leq t \leq d$, and $G(d) = b$. If $f : [a, b] \to X$ is DPB-integrable on $[a, b]$, then $g \cdot (f \circ G)$ is DPB-integrable on $[c, d]$ and $\int_a^b f = \int_c^d g \cdot (f \circ G)$.*

The proof is similar to that of [2] p. 38, 39. Indeed, let us observe that our function g is DPB-integrable (by Tolstov's theorem, [8]). In the diagram on p. 34, [2], we must take X instead of R.

Example. Let $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, $F(0) = 0$ and $w \in X$, $w \neq 0$. The function $f = D(F) \cdot w$ is DPB-intergrable on [0,1], but f is not B-integrable on this interval. This remark is an obvious consequence of a Saks' example, see also [2] p. 35.

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