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RINGS WITH A DERIVATION WHOSE IMAGE IS CONTAINED IN THE NUCLEI

CHEN-TE YEN

Abstract. Let R be a nonassociative ring, N, M, L and G the left nucleus, middle nucleus, right nucleus and nucleus respectively. Suh [4] proved that if R is a prime ring with a derivation d such that $d(R) \subseteq G$ then either R is associative or $d^3 = 0$. We improve this result by concluding that either R is associative or $d^2 = 2d = 0$ under the weaker hypothesis $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. Using our result, we obtain the theorems of Posner [3] and Yen [11] for the prime nonassociative rings. In our recent papers we partially generalize the above main result.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notations for associators and commutators: $(x, y, z) = (xy)z - x(yz)$ and $(x, y) = xy - yx$. We shall denote the left nucleus, middle nucleus, right nucleus and nucleus by N, M, L and G respectively. Thus N, M, L and G consists of all elements n such that $(n, R, R) = 0, (R, n, R) = 0, (R, R, n) = 0$ and $(n, R, R) = (R, n, R) = (R, R, n) = 0$ respectively. An additive mapping d on R is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x, y in R. R is called semiprime if the only ideal of R which squares to zero is the zero ideal. R is called prime if the product of any two nonzero ideals of R is nonzero. R is called simple if R is the only nonzero ideal of R. Clearly, a prime ring is a semiprime ring. If R is a simple ring, then $R^2 = 0$ or $R^2 = R$; in the former case R is commutative and associative. So, if R is a simple ring then we assume that $R^2 = R$. Thus a simple ring is a prime ring. Recently, Suh [4] proved that if R is a prime ring with a derivation d such that $d(R) \subseteq G$ then either R is associative or $d^3 = 0$. In section 2, we improve this result by concluding that either R is associative or $d^2 = 2d = 0$ under the weaker hypothesis

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$d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. Using our result, we obtain the theorems of Posner [3] and Yen [11] for the prime nonassociative rings. In section 3, we partially generalize the main result of this paper and state our recent results. Assume that R has a derivation d. A nonempty subset S of R is called d-invariant if $d(S) \subseteq S$. By the definition of d, we obtain

$$d(R) + d(R)R = d(R) + Rd(R). \tag{1}$$

Rings with associators in the nuclei were first studied by Kleinfeld and later by the author. Kleinfeld [1] proved that if R is a semiprime ring such that $(R, R, R) \subseteq G$ and the Abelian group $(R, +)$ has no elements of order 2 then R is associative. Yen [6] improved this result by dropping the hypothesis $(R, R, R) \subseteq M$. In [5], Yen showed that if R is a simple ring of characteristic not two such that $(R, R, R) \subseteq N \cap M$ or $(R, R, R) \subseteq M \cap L$ then R is associative. For the related results, see [7]-[11].

2. Results and applications

Let R be a nonassociative ring. In every ring one may verify the Teichmüller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \tag{2}$$

Suppose that $n \in N$. Then with $w = n$ in (2) we obtain

$$(nx, y, z) = n(x, y, z) \quad \text{for all } n \text{ in } N. \tag{3}$$

Assume that $m \in L$. Then with $z = m$ in (2) we get

$$(w, x, ym) = (w, x, y)m \quad \text{for all } m \text{ in } L. \tag{4}$$

As consequences of (2), (3) and (4), we have that $N, M, L, N \cap L, N \cap M, M \cap L$ and G are associative subrings of R.

In this section, we assume that R has a derivation d which satisfies

$$(*) \qquad d(R) \subseteq A, \text{ where } A = N \text{ or } A = L.$$

Using $(*)$ and the definition of d, we have

$$d(x)y + xd(y) \in A \quad \text{for all } x, y \text{ in } R. \tag{5}$$

Then with $x \in d(R)$ and $y \in d(R)$ in (5) respectively, and noting that A is an associative subring of R, and using $(*)$ we get

$$d^2(R)R \subseteq A \quad \text{and} \quad Rd^2(R) \subseteq A. \tag{6}$$

Applying $(*)$, (3), (4) and (6), and with $n \in d^2(R)$ in (3), and with $m \in d^2(R)$ in (4) respectively, we obtain

$$d^2(R)(R,R,R) = 0 \quad \text{if} \quad A = N \quad \text{and} \quad (R,R,R)d^2(R) = 0 \quad \text{if} \quad A = L. \tag{7}$$

Combining (7) with $(*)$ yields

$$d^2(R)((R,R,R)R) = 0 \text{ if } A = N \text{ and } (R(R,R,R))d^2(R) = 0 \text{ if } A = L. \tag{8}$$

Definition. The associator ideal I of R is the smallest ideal which contains all associators in R.

Note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (2). Hence we can easily show that

$$I = \sum(R,R,R) + (R,R,R)R = \sum(R,R,R) + R(R,R,R). \tag{9}$$

Using (7), (8) and (9), we obtain

$$d^2(R)\cdot I = 0 \quad \text{if} \quad A = N \quad \text{and} \quad I \cdot d^2(R) = 0 \quad \text{if} \quad A = L. \tag{10}$$

Applying the definition of d, we have the equality

$$d((x,y,z)) = (d(x),y,z) + (x,d(y),z) + (x,y,d(z)). \tag{11}$$

Combining (9) with (11) yields

Lemma 1. *If R is a ring with a derivation d, then the associator ideal I of R is d-invariant.*

Using (1) and the following hypothesis, we can prove

Lemma 2. *If R is a ring with a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$, then the ideal B of R generated by $d(R)$ is $B = \sum d(R) + d(R)R = \sum d(R) + Rd(R)$. Moreover, if we define B_k inductively by $B_1 = B$, and $B_{k+1} = B_k^2$ for every positive integer k, then each $B_k = \sum d(R)^1 + d(R)^i R = \sum d(R)^i + Rd(R)^i$ is an ideal of R, where $i = 2^{k-1}$.*

Lemma 3. *Let R be a semiprime ring with a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. If d^2 is a derivation of R, then $2d = 0$.*

Proof. Since d and d^2 are derivations of R, for all x, y in R we have $d^2(x)y + 2d(x)d(y) + xd^2(y) = d^2(xy) = d^2(x)y + xd^2(y)$. Thus, $2d(x)d(y) = 0$ and so $2d(R)^2 = 0$. Using this, the hypothesis and Lemma 2, we obtain $2(\sum d(R) + d(R)R)^2 = 0$. By Lemma 2 again and the semiprimeness of R, this implies $2(\sum d(R) + d(R)R) = 0$. Hence $2d(R) = 0$, as desired.

Lemma 4. *Let R be a prime ring with a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. If R is not associative and $2d = 0$, then $d^2 = 0$.*

Proof. Since R is not associative, we have $I \neq 0$. Because of $2d = 0$, we obtain

$$d^2(R) + d^2(R)R = d^2(R) + Rd^2(R). \tag{12}$$

Using (12) and the hypothesis, we have that the ideal C of R generated by $d^2(R)$ is $C = \sum d^2(R) + d^2(R)R$. Applying this, (10), (12) and the hypothesis, we get $C \cdot I = 0$. By the primeness of R and $I \neq 0$, this implies $C = 0$. Thus $d^2(R) = 0$, as desired.

Lemma 5. *If R is a prime ring with a derivation d such that $d(R) \subseteq N \cap L$, then either R is associative or $d^2 = 2d = 0$.*

Proof. If $I = 0$, then R is associative. Assume that $I \neq 0$, and $x, y, z \in R$ and $t \in I$. By (10), we have $d^2(R) \cdot I = 0$. Using this and $d(R) \subseteq N$, we get $0 = d^2(xy) \cdot t = (d^2(x)y + 2d(x)d(y) + xd^2(y))t = d^2(x) \cdot yt + 2d(x)d(y) \cdot t + xd^2(y) \cdot t = 2d(x)d(y) \cdot t + xd^2(y) \cdot t$ and so

$$2d(x)d(y) \cdot t = -xd^2(y) \cdot t \quad \text{for all} \quad x, y \in R \quad \text{and} \quad t \in I. \tag{13}$$

By Lemma 1, we have $d(I) \subseteq I$. Thus replacing t by $d(t)$ in (13), and applying $d(R) \subseteq L$, Lemma 1 and $d^2(R) \cdot I = 0$, we obtain $2d(x)d(y) \cdot d(t) = -xd^2(y) \cdot d(t) = -x \cdot d^2(y)d(t) = 0$. Hence, we get

$$2d(R)^2 \cdot d(I) = 0. \tag{14}$$

Using $d(R) \subseteq N \cap L$ and (14), we have

$$\begin{aligned} 2d(x)d(y) \cdot zd(t) &= 2(d(x)d(y) \cdot z)d(t) = 2(d(x)(d(yz) - yd(z)))d(t) \\ &= 2d(x)d(yz) \cdot d(t) - 2(d(x)y \cdot d(z))d(t) = -2((d(xy) - xd(y))d(z))d(t) \\ &= -2d(xy)d(z) \cdot d(t) + 2(x \cdot d(y)d(z))d(t) = x \cdot (2d(y)d(z))d(t) = 0. \end{aligned}$$

Applying this, $d(R) \subseteq N \cap L$ and (14), we obtain $2d(x)d(y)d(z) \cdot t = 2d(x)d(y) \cdot d(z)t = 2d(x)d(y)(d(zt) - zd(t)) = 2d(x)d(y) \cdot d(zt) - 2d(x)d(y) \cdot zd(t) = 0$. Thus, we get

$$2d(R)^3 \cdot I = 0. \tag{15}$$

Using Lemma 2 and (15), we have $2((\sum d(R) + d(R)R)^2)^2 \cdot I = 0$. By the primeness of R, and applying $I \neq 0$ and Lemma 2 twice, this implies $2((\sum d(R) + d(R)R)^2)^2 = 0$ and so $2(\sum d(R) + d(R)R)^2 = 0$. Again, the last equality implies $2(\sum d(R) + d(R)R) = 0$. Hence, $2d(R) = 0$. By Lemma 4 and $I \neq 0$, we obtain $d^2 = 0$. This completes the proof of Lemma 5.

Lemma 6. *If R is a prime ring with a derivation d such that $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$, then either R is associative or $d^2 = 2d = 0$.*

Proof. By symmetry, we only prove the lemma in case $d(R) \subseteq N \cap M$. If $I = 0$, then R is associative. Assume that $I \neq 0$. By (10), we have $d^2(R) \cdot I = 0$. Using this and $d(R) \subseteq N \cap M$, for all $x, y \in R$ and $z \in I$ we get
$0 = d^2(xy)z = (d^2(x)y + 2d(x)d(y) + xd^2(y))z = d^2(x)(yz) + 2(d(x)d(y))z + (xd^2(y)))z = 2(d(x)d(y))z + x(d^2(y)z) = 2(d(x)d(y))z.$

Hence, we obtain $2d(R)^2 \cdot I = 0$. Applying this, $d(R) \subseteq N \cap M$ and Lemma 2, we have $2(\sum d(R) + d(R)R)^2 \cdot I = 0$. By the primeness of R, and using $I \neq 0$ and Lemma 2 twice, this implies $2(\sum d(R) + d(R)R)^2 = 0$ and so $2(\sum d(R) + d(R)R) = 0$. Thus, $2d(R) = 0$. Because of $I \neq 0$, by Lemma 4 we get $d^2 = 0$. This completes the proof of Lemma 6.

Combining Lemma 5 with Lemma 6 yields the main result of this paper.

Theorem 1. *If R is a prime ring with a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$, then either R is associative or $d^2 = 2d = 0$.*

Corollary 1. *If R is a prime ring of characteristic not two with a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$, then either R is associative or $d = 0$.*

In the courses of the proofs of Lemma 5 and Lemma 6, we obtain

Corollary 2. *If R is a semiprime ring with a derivation d such that $d(R) \subseteq N \cap L \cap I$ or $d(R) \subseteq N \cap M \cap I$ or $d(R) \subseteq M \cap L \cap I$, then $d^2 = 2d = 0$.*

Corollary 3. *If R is a semiprime ring such that the Abelian group $(R, +)$ has no elements of order 2 and R has a derivation d such that $d(R) \subseteq N \cap L \cap I$ or $d(R) \subseteq N \cap M \cap I$ or $d(R) \subseteq M \cap L \cap I$, then $d = 0$.*

Applying Theorem 1, we can generalize the results of prime associative rings of characteristic not two with a derivation to the prime nonassociative rings. Here, we give two applications. The first application of Theorem 1 is by using Theorem 1 of [3] to obtain the theorem of Posner for the prime nonassociative rings.

Theorem 2. *Let R be a prime ring of characteristic not two with derivations d and f such that $g(R) \subseteq N \cap L$ or $g(R) \subseteq N \cap M$ or $g(R) \subseteq M \cap L$, where $g = d$ or $g = f$. If fd is a derivation of R, then either $d = 0$ or $f = 0$.*

The second application of Theorem 1 is by applying the theorem of [11] to obtain this result for the prime nonassociative rings.

Theorem 3. *Let R be a noncommutative prime ring of characteristic not two with a nonzero derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. Then the subring of R generated by all $(d(x), y), x, y \in R$ contains a nonzero two-sided ideal of R.*

Using Lemma 3, Theorem 2 is valid for the semiprime ring case when $f = d$ and the Abelian group $(R, +)$ has no elements of order 2.

Theorem 4. *Let R be a semiprime ring such that the Abelian group $(R, +)$ has no elements of order 2 and let R have a derivation d such that $d(R) \subseteq N \cap L$ or $d(R) \subseteq N \cap M$ or $d(R) \subseteq M \cap L$. If d^2 is a derivation of R, then $d = 0$.*

3. Partial generalizations

Recently, we partially generalize Theorem 1.

Theorem 5 [8]. *If R is a simple ring with a derivation d and there exists a fixed positive integer n such that $d^n(R) \subseteq N \cap L$, then either R is associative or $d^{3n-1} = 0$.*

Theorem 6 [8]. *If R is a prime ring with a derivation d and there exists a fixed positive integer n such that $d^n(R) \subseteq G$, then either R is associative or $d^{3n-1} = 0$.*

Theorem 5 remains true for the prime ring case by adding the hypothesis $d^{3n-1}(R) \subseteq M$. Thus this result extends Theorem 6 and partially generalizes Theorem 1. For the proof, we need a lemma.

Lemma 7 [10]. *Let R be a ring and E a nonempty subset of G. If $RE \subseteq N$ and $ER \subseteq L$, or $ER + RE \subseteq M$, then $ER + RE \subseteq M$, and the ideal F of R generated by E is $F = \sum E + ER + RE + R \cdot ER$.*

Theorem 7. *If R is a prime ring with a derivation d and there exists a fixed positive integer n such that $d^n(R) \subseteq N \cap L$, and $d^{3n-1}(R) \subseteq M$, then either R is associative or $d^{3n-1} = 0$.*

Proof. By the hypothesis, we get $d^{3n-1}(R) \subseteq G$. Using $d^n(R) \subseteq N \cap L$, and as the proofs of the results of [8], we have

$$d^{3n-1}(R)R + Rd^{3n-1}(R) \subseteq N \cap L \quad \text{and} \quad d^{3n-1}(R) \cdot I = 0. \tag{16}$$

Applying (16), $d^{3n-1}(R) \subseteq G$ and Lemma 7, we obtain that $d^{3n-1}(R)R + Rd^{3n-1}(R) \subseteq M$, and the ideal K of R generated by $d^{3n-1}(R)$ is $K = \sum d^{3n-1}(R) + d^{3n-1}(R)R + Rd^{3n-1}(R) + R \cdot d^{3n-1}(R)R$. Using these, $d^{3n-1}(R) \subseteq G$ and (16), we get $K \cdot I = 0$. By the primeness of R, this implies $K = 0$ or $I = 0$. If $I = 0$, then R is associative. Assume that $K = 0$. Then $d^{3n-1}(R) = 0$. This completes the proof of Theorem 7.

By an argument similar to the proof of Theorem 7, we can show the following result which also generalizes Theorem 6 and partially extends Theorem 1.

Theorem 8. *If R is a prime ring with a derivation d and there exists a* *fixed positive integer* n *such that* $d^n(R) \subseteq N \cap M$, *(resp.* $d^n(R) \subseteq M \cap L$*) and* $d^{3n-1}(R) \subseteq L$ *(resp.* $d^{3n-1}(R) \subseteq N$*), then either* R *is associative or* $d^{3n-1} = 0$.

In Theorem 7, without the hypothesis $d^{3n-1}(R) \subseteq M$ we obtain

Theorem 9 [8]. *If* R *is a prime ring with a derivation* d *and there exists a fixed positive integer* n *such that* $d^n(R) \subseteq N \cap L$, *then either* R *is associative or* $d^{3n-1}(R)^2 = 0$.

Recently, using Theorem 1 of [2] we also partially extends Theorem 1.

Theorem 10 [7]. *If* R *is a prime ring with a derivation* d *and there exists a fixed positive integer* n *such that* $d^n(R) \subseteq G$ *and* $(d^n(R), R) = 0$, *then* R *is associative and* $d^n = 0$, *or* R *is associative and commutative, or*

$$d^{2n} = (\frac{(2n)!}{n!})d^n = 0.$$

Added in proof. Recently, we have proved that if R is a semiprime ring such that $(R,R,R) \subseteq N \cap L$ or $(R,R,R) \subseteq N \cap M$ or $(R,R,R) \subseteq M \cap L$ then $N = M = L$. Thus E. Kleinfeld's result [1] can be improved. We also have proved that if R is a semiprime ring with a derivation d such that $d(R) \subseteq G$ then $d^2(I) = 2d(I) = 0$.

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Department of Mathematics, Chung Yuan University, Chung Li, Taiwan, 320, Republic of China.