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SOME THEOREMS ON A GENERALIZED LAPLACE TRANSFORM OF GENERALIZED FUNCTIONS

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Abstract. In this paper we extend the generalized Laplace transform $F(s) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \int_0^\infty (st)^\beta \; {}_1F_1(\beta+\eta+1, \alpha+\beta+\eta+1; -st)f(t)dt$ where $f(t) \in L(0,\infty)$, $\beta \geq 0$, $\eta > 0$; to a class of generalized functions. We will extend the above transform to a class of generalized functions as a special case of the convolution transform and prove an inversion formula for it.

1. Introduction

A generalization of the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t)dt \tag{1.1}$$

is given by Joshi [5]:

$$F(s) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+1)} \int_0^\infty (st)^\beta \; {}_1F_1(\beta+\eta+1, \alpha+\beta+\eta+1; -st)f(t)dt. \tag{1.2}$$

Recently Gupta and Mahato [2] extended the transform (1.2) to a class of generalized functions and an analyticity theorem is proved for it. In Gupta and Mahato [3], a complex inversion formula for (1.2) has been extended to a class of generalized functions.

In this paper we discuss the generalized Laplace transform (1.2) as a special case of the convolution transform

$$F(x) = \langle f(t), G(x-t)\rangle. \tag{1.3}$$

We also prove an inversion formula for (1.2) in the distributional sense.

2. The testing function spaces $D, D(I), L_{c,d}, J_{c,d}$ and their duals

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A function is said to be smooth if its derivatives of all orders are continuous at all points of its domain. The space D consists of all complex-valued functions ϕ that are smooth zero outside some finite interval. Let I be the interval $0 < t < \infty$. $D(I)$ is the space of infinitely differentiable functions having compact supports defined over the interval I.
Let $\lambda_{c,d}(t)$ be the function:

$$\lambda_{c,d}(t) = \begin{cases} e^{ct}, & 0 \leq t < \infty \\ e^{dt}, & -\infty < t < 0. \end{cases} \tag{2.1}$$

$L_{c,d}$ denotes the space of all complex-valued smooth functions $\phi(t)$ on $-\infty < t < \infty$ on which the functionals defined by

$$\gamma_k(\phi) \stackrel{\triangle}{=} \gamma_{c,d,k}(\phi) = \sup_{-\infty<t<\infty} |\lambda_{c,d}(t) D^k \phi(t)| \tag{2.2}$$

assume finite values. We assign to $L_{c,d}$ the topology generated by $\left\{\gamma_k\right\}_{k=0}^{\infty}$ thereby making it a countably multinormed space.

Applying the change of variable

$$T = e^{-t},\ t = (-\log T),\ D_t = (-T\, D_T)$$

to the definition of $L_{c,d}$ and setting $T\psi(T) = \phi(-\log T)$ in (2.2), we have the following definition:

Given any two real numbers c and d, $J_{c,d}$ is the space of all smooth functions $\psi(T)$ on $0 < T < \infty$ such that

$$\begin{aligned} i_k\left\{\psi(T)\right\} &\stackrel{\triangle}{=} i_{c,d,k}\left\{\psi(T)\right\} \\ &= \sup_{0<T<\infty} \left|\lambda_{c,d}(-\log T)(-TD_T)^k \left\{T\psi(T)\right\}\right| \\ &< \infty,\ k = 0, 1, 2, \ldots \end{aligned} \tag{2.3}$$

where

$$\lambda_{c,d}(-\log T) = \begin{cases} T^{-c}, \ 0 < T \leq 1 \\ T^{-d}, \ 1 < T < \infty. \end{cases} \tag{2.4}$$

The topology of $J_{c,d}$ is that generated by the multinorm $\left\{i_{c,d,k}\right\}_{k=0}^{\infty}$. As a consequence $J_{c,d}$ is a complete countably multinormed space.

$D', D'(I), L'_{c,d}$ and $J'_{c,d}$ are the dual spaces corresponding to the spaces $D, D(I), L_{c,d}$ and $J_{c,d}$ respectively.

Theorem 2.1. *The mapping*

$$\psi(T) \mapsto e^{-t}\psi(e^{-t}) = \phi(t) \tag{2.5}$$

is an isomorphism from $J_{c,d}$ *onto* $L_{c,d}$.

The inverse mapping is given by

$$\phi(t) \mapsto (T^{-1})\phi(-\log T) = \psi(T) \tag{2.6}$$

Proof. It is obvious that the mapping (2.5) and (2.6) are linear and inverses of one another.

Let $\psi(T) \in J_{c,d}$. Some computations show that $D_t^k\left\{e^{-t}\psi(e^{-t})\right\}$ is equal to a finite sum of terms, a typical term being $a_p T^{p+1} D_T^p \psi(T)$, where $0 \leq p \leq k$ and a_p is a constant. Thus,

$$\begin{aligned}
&\lambda_{c,d}(t) D_t^k\left[e^{-t}\psi(e^{-t})\right] \\
&= \sum_p a_p \lambda_{c,d}(-\log T) T^{p+1} D_T^p \psi(T) \\
&= \sum_p b_p \lambda_{c,d}(-\log T)(-T D_T)^p \left\{T\psi(T)\right\}
\end{aligned} \tag{2.7}$$

(where b_p is another constant) so that

$$\begin{aligned}
\gamma_{c,d,k}(\phi) &= \gamma_{c,d,k}\left[e^{-t}\psi(e^{-t})\right] \\
&\leq \sum_p |b_p| i_{c,d,p}\left\{\psi(T)\right\}.
\end{aligned} \tag{2.8}$$

Consequently, (2.5) is a continuous mapping of $J_{c,d}$ into $L_{c,d}$. Now, let $\phi(t) \in L_{c,d}$. Again a straightforward computation shows that

$$(-T D_T)^k [T T^{-1} \phi(-\log T)] = \sum_p C_p D_t^p \phi(t)$$

where $0 \leq p \leq k$ and the $C_p{}'s$ are constants.

Therefore,

$$\begin{aligned}
i_{c,d,k}(\psi) &= i_{c,d,k}\left[T^{-1}\phi(-\log T)\right] \\
&\leq \sum_p |C_p| \gamma_{c,d,p}(\phi).
\end{aligned}$$

Thus (2.6) is a continuous linear mapping of $L_{c,d}$ into $J_{c,d}$. Since the mapping (2.5) and (2.6) are one to one we can now conclude that they are also onto $L_{c,d}$ and $J_{c,d}$ respectively. Our proof is complete.

The dual space $L'_{c,d}$ denotes the space of continuous linear functions on $L_{c,d}$. If $f(f) \in L'_{c,d}$ we define $f(-\log T)$ as a functional on $J_{c,d}$ by

$$\langle f(-\log T), T^{-1}\phi(-\log T)\rangle \triangleq \langle f(t), \phi(t)\rangle, \quad \phi(t) \in L_{c,d}. \tag{2.9}$$

It can be easily proved that the mapping $f(t) \mapsto f(-\log T)$ defined by (2.9) is an isomorphism on $L'_{c,d}$ onto $J'_{c,d}$. The inverse mapping $f(T) \mapsto f(e^{-t})$ is defined by

$$\langle f(e^{-t}), \phi(t)\rangle \triangleq \langle f(T), \psi(T)\rangle \tag{2.10}$$

3. The generalized one-sided Laplace transformation

Let $G(t) = \frac{\Gamma(A)}{\Gamma(B)} e^{\beta t}\ {}_1F_1(A, B, e^{-t})e^t$ where $A = \beta + \eta + 1; B = \alpha + \beta + \eta + 1$. Setting $y = e^x, T = e^{-t}$ and $\phi(t) = G(x-t)$, we obtain

$$\begin{aligned} T^{-1}\phi(-\log T) =& e^t G(x-t) \\ =& \frac{\Gamma(A)}{\Gamma(B)} y(yT)^\beta\ {}_1F_1(A, B; -yT). \end{aligned}$$

If we choose $c < 1$ and d as any real number, we may replace $\phi(t)$ by $G(x-t)$ in (2.9) to obtain

$$\begin{aligned} &\langle f(-\log T), \frac{\Gamma(A)}{\Gamma(B)} y(yT)^\beta\ {}_1F_1(A, B; -yT)\rangle \\ =&\langle f(t), G(x-t)\rangle \\ =&F(\log y). \end{aligned}$$

Setting $J(y) = y^{-1}G(\log y)$ and $j(T) = f(-\log T)$ we finally obtain the new definition of the generalized one-sided Laplace transform

$$J(y) \triangleq \langle j(T), \frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_2F_1(A, B; -yT)\rangle, 0 < y < \infty. \tag{2.11}$$

This has a meaning as the application of $j(T) \in J'_{c,d}$ to $\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\, {}_2F_1(A, B; -yT) \in J_{c,d}$ where $c < 1$ and d is arbitrary and positive.

Theorem 3.1. *For any fixed real $y > 0$, $\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_1F_1(A, B; -yT)$ is a member of $J_{c,d}$.*

Proof. For $\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_1F_1(A, B; -yT)$ to be in $J_{c,d}$ we have to show that

$$\sup_{0<T<\infty} \left|\lambda_{c,d}(-\log T)(-TD_T)^k\left\{T\psi(T)\right\}\right|$$

is bounded, where $\psi(T) = \frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_1F_1(A, B; -yT)$.
Now

$$\begin{aligned} &\sup_{0<T<\infty} \left|\lambda_{c,d}(-\log T)(-TD_T)^k\left\{T\psi(T)\right\}\right| \qquad (2.12) \\ =& \sup_{0<T<\infty} \left|\lambda_{c,d}(-\log T)(-1)^k \sum_p T^{p+\beta+1} D_T^p\left\{-\frac{\Gamma(A)}{\Gamma(B)}\ {}_1F_1(A, B; -yT)\right\}\right|,\ 0 \le p \le k. \end{aligned}$$

By using the result of Erdelyi [1] p.254

$$\frac{d^n}{dx^n}[\ {}_1F_1(a;c;x)] = \frac{(a)_n}{(c)_n}\ {}_1F_1(a+n,c+n;x)$$

where $(a)_n = a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$.
Thus each side of (2.12) is equal to

$$\sup_{0<T<\infty}\left|\lambda_{c,d}(-\log T)(-1)^k \sum_p (-1)^p T^{p+\beta+1} y^p \frac{\Gamma(A+p)}{\Gamma(B+p)}\ {}_1F_1(A+p,B+p;-yT)\right|. \quad (2.13)$$

For $0 < T \leq 1$, the above expression is

$$\sup_{0<T\leq 1}\left|\sum_p T^{-c+p+\beta+1}(-1)^{k+p} y^p \frac{\Gamma(A+p)}{\Gamma(B+p)}\ {}_1F_1(A+p,B+p;-yT)\right|.$$

From Slater [6], p.59, we have

$${}_1F_1(a,b;-x) = \frac{\Gamma(b)}{\Gamma(b-a)} x^{-a}\left\{1 + O\,|x|^{-1}\right\},\ x \to \infty$$

and

$${}_1F_1(a,b;-x) = O(1),\ x \to 0.$$

Thus for $0 < T \leq 1$ (2.13) is

$$= \sup_{0<T\leq 1}\left|\sum_p T^{-c+p+\beta+1}(-1)^{k+p} y^p \frac{\Gamma(A+p)}{\Gamma(B+p)}\right|$$

=a finite quantity as $|T| \to 0$ if $1-c>0$, i.e. $c<1$.

Now for $T \to \infty$, (2.13) is

$$= \sup_{0<T<\infty}\left|\sum_p T^{-d+p+\beta+1}(-1)^{k+p} y^p \frac{\Gamma(A)}{\Gamma(B)}\frac{\Gamma(\beta)}{\Gamma(\alpha)}(yT)^{-\beta-x-1-p}\right|$$

$$= \sup_{0<T<\infty}\left|\sum \frac{\Gamma(A)}{\Gamma(\alpha)}(-1)^{k+p} T^{-d-n} y^{-\beta-\eta-1}\right|$$

=a finite quantity for any real positive value of d.

Thus, $\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_1F_1(A,B;-yT)$ is a member of $J_{c,d}$ for every $c<1$ and $d>0$.
It can be easily proved that $J(y)$ is a smooth function on $0<y<\infty$. Thus if

$$J(y) = \langle j(T), -\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta\ {}_1F_1(A,B;-yT)\rangle$$

then

$$J^{(n)}(y) = \left\langle j(T), \frac{\partial^h}{\partial y^h}\left\{-\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta \ {}_1F_1(A,B;-yT)\right\}\right\rangle$$

where n is a non-negative integer.

Theorem 3.2. *Let $j(T)$ be an arbitrary element of $J'_{c,d}$ and*

$$J(y) = \left\langle j(T), -\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta \ {}_2F_1(A,B;-yT)\right\rangle.$$

Then

$$j(y) = O[y^{-p-\eta-1}] \quad as \quad y \to \infty.$$

Proof. Using the boundedness property of generalized functions we get

$$J(y) \leq C \max_{0<M<V} \sup_{0<T<\infty} \left|\lambda_{c,d}(-\log T)(-TD_T)^M\left\{T\psi(T)\right\}\right|$$

where $\psi(T) = \frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta \ {}_1F_1(A,B;-yT)$.
For $1 < T < \infty$,

$$\begin{aligned}
&\lambda_{c,d}(-\log T)(-TD_T)^M\left\{T\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta \ {}_1F_1(A,B;-yT)\right\}\\
=&T^{-d}(-TD_T)^M\left\{T\frac{\Gamma(A)}{\Gamma(B)}(yT)^\beta \ {}_1F_1(A,B;-yT)\right\}\\
=&T^{-d}(-1)^M \sum_{0\leq p<M} a_pT^{\beta+p+1}D_T^p\left\{\frac{\Gamma(A)}{\Gamma(B)} \ {}_1F_1(A,B;-yT)\right\}(a_p \text{ being constant})\\
=&(-1)^MT^{-d}[\sum_p a_pT^{p+\beta+1}(-1)^py^p\frac{\Gamma(A+p)}{\Gamma(B+p)} \ {}_1F_1(A+p,B+p;-yT)]\\
=&\sum_p(-1)^{M+p}a_pT^{-d+p+\beta+1}y^p\frac{\Gamma(A+p)}{\Gamma(B+p)} \ {}_1F_1(A+p,B+p;-yT)\\
=&\sum_p(-1)^{M+p}a_pT^{-d+p+\beta+1}y^p\frac{\Gamma(A+p)}{\Gamma(B+p)}\frac{\Gamma(B+p)}{\Gamma(\alpha)}(yT)^{-\beta-\eta-1-p}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(Slater [6], p.59)}\\
=&\sum_p(-1)^{M+p}T^{-d-n}y^{-\beta-\eta-1}a_p\frac{\Gamma(A+p)}{\Gamma(\alpha)}.
\end{aligned}$$

For large T and $y > 0$, we have $t\,T^{-d-\eta}y^{-\beta-\eta-1} \leq$ (const.) $y^{-\beta-\eta-1}$. This establishes that for large T,

$$|J(y)| \leq (\text{const.})\, y^{-\beta-\eta-1}.$$

So $J(y) = O[y^{-\beta-\eta-1}]$ as $y \to \infty$.

4. Inversion formula

The technique employed in finding the inversion formula is as given in Zemanian [7] p.229-226.

The conventional convolution transform is

$$F(x) = \int_{-\infty}^{\infty} f(t)G(x-t)dt \tag{4.2}$$

and the corresponding inversion function $E(x)$, which serves to invert the transform is defined by the equation

$$[E(x)]^{-1} = \int_{-\infty}^{\infty} G(y)e^{-xy}dy.$$

This conventional convolution has been extended a certain class of generalized functions [Zemanian, 7, p.229-246] and their inversion formula has been proved to be still valid when the limiting operation in that formula is understood as weak convergence in the space D' of Schwartz distributions.

Setting $s = e^x$ and $t = e^{-t}$ in the conventional generalized Laplace transform

$$F(s) = \frac{\Gamma(A)}{\Gamma(B)} \int_0^{\infty} (st)^{\beta}\ {}_1F_1(A, B; -st)f(t)dt.$$

We obtain

$$\begin{aligned} F(e^x) =& \frac{\Gamma(A)}{\Gamma(B)} \int_{-\infty}^{\infty} e^{(x-t)\beta}\ {}_1F_1(A, B; -e^{(x-t)})f(e^{-t})e^{-t}dt \\ \text{or} \quad e^x F(e^x) =& \frac{\Gamma(A)}{\Gamma(B)} \int_{-\infty}^{\infty} e^{(x-t)(\beta+1)}\ {}_1F_1(A, B; -e^{(x-t)})f(e^{-t})dt \\ \text{or} \quad J(x) =& \frac{\Gamma(A)}{\Gamma(B)} \int_{-\infty}^{\infty} e^{(x-t)(\beta+1)}\ {}_1F_1(A, B; -e^{(x-t)})j(t)dt \\ \text{where} \quad J(x) =& e^x F(e^x) \text{ and } j(t) = f(e^{-t}). \end{aligned}$$

As in Joshi [5] and using some results from Hirschman and Widder [4] (p.66), the inversion operator $E(D)$ is given by

$$\begin{aligned} &E(D)\Big\{e^x F(e^x)\Big\} = E(D)\Big\{J(x)\Big\} \\ &= \lim_{n\to\infty} (-1)^n n^{\alpha-\beta+x} e^{(n+\beta)x} D_1^n e^{-(\beta+\eta+1)x} D_1^n e^{-\alpha x} D_1^{-n} e^{(\alpha+\eta)x} F(e^x) \\ &= f(e^{-t}) \end{aligned}$$

where $D_1 = \frac{d}{de^x}$.

Returning to the original variable, we have

$$\lim_{n\to\infty}(-1)^n\frac{\Gamma(n+\alpha)n^{\log x}}{\Gamma(n+\beta)(n+1)}s^{n+\beta}D^nS^{-(\beta+\eta+1)}D^ns^{-\alpha}D^{-n}s^{\alpha+\eta}F(s)=f(t).$$

This has a sense as a limit in $D'(I)$. The change of variable we have used in (2.11) defines an isomorphism from D' onto $D'(I)$.

In summary if $j\in J'_{c,d}$ for some $c<1$ and $d>0$ and if $J(x)$ is defined by (2.11), (4.3) holds true in the sense of convergence in $D'(I)$.

As a consequence of inversion formula, we have Theorem 4.1 (The Uniqueness Theorem).

Let $f\in L'_{c,d}$ and $h\in L'_{c,d}$. Also let $F(s)=\langle f(t),G(s-t)\rangle$ and $H(s)=\langle h(t),G(s-t)\rangle$. If $F(x)=H(x)$ for all x, then $f=h$ in the sense of equality in D'.

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