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OSCILLATION AND COMPARISON THEOREMS FOR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. In this paper we study qualitative properties of solutions of the neutral difference equation

$$\begin{cases} \Delta(y_n - py_{n-k}) + \sum_{i=1}^{m} q_n^i y_{n-k_i} = 0 \\ y_n = A_n \quad for\ n = -M, \dots, -1, 0 \end{cases}$$

where $p \geq 1$, $M = \max\{k, k_1, \dots, k_m\}$, and $k, k_i, i = 1, \dots, m$, are nonnegative integers. Riccati techniques are used.

Introduction

In a number of recent papers [2-10], the oscillation and nonoscillation of solutions of delay difference equations are being extensively investigated. In paticular the oscillation of solutions of the neutral difference equation

$$\Delta(y_n - cy_{n-k}) + p_n y_{n-m} = 0 \tag{1.1}$$

has been investigated in [3, 6, 8, 9, 10], where $p_n > 0$, $c \in (0,1)$, Δ denotes the forward difference operator $\Delta y_n = y_{n+1} - y_n$. Equation (1.1) was considered by Brayron and Willoughby [1] from the numerical analysis point of view.

In this paper we consider the case that $c \geq 1$ in (1.1), or the equations which we will consider are neutral difference equations of the form

$$\Delta(y_n - py_{n-k}) + \sum_{i=1}^{m} q_n^i y_{n-k_i} = 0, \ n = 0, 1, 2, \dots \tag{1.2}$$

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Let $M = \max\{k, k_1, \dots, k_m\}$, $I_m = \{1, \dots, m\}$, $\tilde{k} = \max_{\{i \in I_m\}} k_i$. where k, k_i are nonnegative integers $i = 1, \dots, m$ and $p \geq 1$.

By a solution of (1.2) we mean a sequence $\{y_n\}$ which is defined for $n \geq -M$ and satisfies equation (1.2) for $n = 0, 1, 2, \dots$, clearly, if

$$y_n = A_n, \text{ for } n = -M, \dots, -1, 0 \tag{1.3}$$

are given, then equation (1.2) has a unique solution satisfying the initial condition (1.3). A nontrivial solution $\{y_n\}$ of equation (1.2) is said to be oscillatory if for every $N > 0$ there exists an $n \geq N$ such that $y_n y_{n+1} \leq 0$, otherwise it is nonoscillatory. In this paper sufficient conditions for all solutions of (1.2) to be oscillatory and (1.2) to have a nonoscillatory solution are obtained respectively. As a consequence we prove that the oscillation of equation (1.2) with periodic coefficients is equivalent to the equation with constant coefficients. Finally, a comparison result for the oscillation of equation (1.2) is derived.

2. Main Results

We assume through out this paper that $(H)\ \sum_{i=1}^{m} q_n^i$ can not be identically zero on $[N_1, N_2]$ with $N_1 < N_2$ where N_1, N_2 are any two positive integers

Lemma 2.1. *Assume that $p \geq 1$, $q_n^i \geq 0$, $i \in I_m$ and*

$$\sum_{j=N}^{\infty} \sum_{i=1}^{m} q_j^i = \infty. \tag{2.1}$$

Let $\{y_n\}$ be an eventually positive solution of (1.2). *Then $z_n < 0$ and $\Delta z_n < 0$ eventually, where*

$$z_n = y_n - p y_{n-k}. \tag{2.2}$$

Proof. From (1.2), $\Delta z_n < 0$. If $z_n \geq 0$, then $y_n \geq p y_{n-k}$ which implies that there exists an $d > 0$ such that $y_n \geq d > 0$ for all large n. Hence from (1.2)

$$\Delta z_n + d \sum_{i=1}^{m} q_n^i \leq 0. \tag{2.3}$$

(2.1) and (2.3) lead to that $z_n \to -\infty$ as $n \to \infty$, a contradiction. Therefore $z_n < 0$ eventually. The proof is completed.

Lemma 2.2. *In addition to assumptions of Lemma 2.1, we further assume that*

$$q_n^i \leq q_{n-k}^i, \quad i \in I_m, \quad n = N, N+1, \dots. \tag{2.4}$$

Let $\{y_n\}$ be an eventually positive solution of (1.2) *and z_n is defined by (2.2). Set*

$$w_n = \frac{\Delta z_n}{z_n} > 0. \tag{2.5}$$

The eventually

$$w_n \geq \frac{1}{p} w_{n+k} \prod_{i=n}^{n+k-1} (1 + w_i) + \frac{1}{p} \sum_{i=1}^{m} q_n^i \prod_{i=n}^{n+k-k_i-1} (1 + w_i). \tag{2.6}$$

Proof. From (2.1) and (2.2) we have

$$\begin{aligned}
\Delta z_n &= -\sum_{i=1}^{m} q_n^i y_{n-k_i} \\
&= -\sum_{i=1}^{m} q_n^i (y_{n-k_i} - p y_{n-k_i-k} + p y_{n-k_i-k}) \\
&= -\sum_{i=1}^{m} q_n^i z_{n-k_i} - p \sum_{i=1}^{m} q_n^i y_{n-k_i-k} \\
&\geq -\sum_{i=1}^{m} q_n^i z_{n-k_i} - p \sum_{i=1}^{m} q_{n-k}^i y_{n-k_i-k} \\
&= -\sum_{i=1}^{m} q_n^i z_{n-k_i} + p \Delta z_{n-k}.
\end{aligned}$$

Exchanging terms in the above inequality we obtain

$$\Delta z_n \leq \frac{\Delta z_{n+k}}{p} + \frac{1}{p} \sum_{i=1}^{m} q_{n+k}^i z_{n+k-k_i}$$

$$\frac{\Delta z_n}{z_n} \geq \frac{\Delta z_{n+k}}{p z_n} + \frac{1}{p z_n} \sum_{i=1}^{m} q_{n+k}^i z_{n+k-k_i}. \tag{2.7}$$

By (2.5), $z_{n+1}/z_n = 1 + w_n$ and hence

$$\frac{z_{n+k}}{z_n} = (1 + w_{n+k-1}) \dots (1 + w_n). \tag{2.8}$$

Substituting (2.5) into (2.7) we obtain (2.6). The proof is finished.

Remark 2.1. If $m = 1$ and $q_n > 0$ for all large n, then (2.4) is not necessary. (2.6) is replaced by

$$w_n \geq \frac{q_n}{p q_{n+k}} w_{n+k} \prod_{i=n}^{n+k-1} (1 + w_i) + \frac{q_n}{p} \prod_{i=n}^{n+k-k_1-1} (1 + w_i). \tag{2.9}$$

Theorem 2.1. *Assume that*

(1) $p > 1, k \geq \tilde{k} + 2;$

(2) $q_n^i \geq 0, n = N-k, N-k+1, \ldots, q_n^i \leq q_{n-k}^i, \mathbf{i} \in I_m, n = N, N+1, \ldots;$

$$\liminf_{n\to\infty} \sum_{j=n}^{n+[\frac{k-\tilde{k}}{2}]-1} (\sum_{i=1}^{m} q_j^i) > 0; \tag{2.10}$$

(3)

$$\inf_{n\geq N} \inf_{\mu>0} \min_{T=k,k-k_i,i\in I_m} \{\frac{1}{p}(1+\mu)^k + \frac{1}{p\mu}\sum_{i=1}^{m}(\frac{1}{T}\sum_{j=n}^{n+T-1} q_{j+k}^i)(1+\mu)^{k-k_i}\} > 1.$$

Then every solution of (1.2) *is oscillatory.*

Proof. Suppose the contrary. Let $\{y_n\}$ be a positive solution of (1.2). It is easy to see that (2.10) implies (2.1). Then Lemma 2.2 holds, i.e., (2.6) holds. Define sequence $\{\lambda_n^{(l)}\}, n = N, N+1, \ldots, l = 1, 2, \ldots$ as follows:

$$\{\lambda_n^{(l)}\} = \{0\}, n = N, N+1, \ldots,$$

$$\lambda_n^{(l+1)} = \frac{1}{p}\lambda_{n+k}^{(l)} \prod_{i=n}^{n+k-1}(1+\lambda_i^{(l)}) + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \prod_{i=n}^{n+k-k_i-1}(1+\lambda_i^{(l)}). \tag{2.11}$$

$$n = N, N+1, \ldots, l = 1, 2, \ldots$$

Define a sequence of numbers as follows:

$$\mu_1 = 0$$

$$\mu_{r+1} = \inf_{n\geq N} \min_{T=k,k-k_i,i\in I_m} [\frac{1}{p}\mu_r(1+\mu_r)^k + \frac{1}{p}\sum_{i=1}^{m}(\frac{1}{T}\sum_{j=n}^{n+T-1} q_{j+k}^i)(1+\mu_r)^{k-k_i}]$$

$$r = 1, 2, \ldots \tag{2.12}$$

Condition (3) implies that

$$0 = \mu_1 < \mu_2 < \ldots.$$

(2.6) implies that

$$\lambda_n^{(l+1)} \leq w_n, \ l = 0, 1, 2, \ldots, \ n = N, N+1, \ldots$$

and

$$\frac{1}{T}\sum_{j=n}^{n+T-1} \lambda_n^{(l+1)} \geq \mu_{l+1}, \ l = 0, 1, 2, \ldots \tag{2.13}$$

where $T = k, k - k_i, i \in I_m, n = N, N+1, \dots$ From (2.12) and condition (3), if $\mu^* = \lim_{k\to\infty} \mu_k$ exists and μ^* is finite, then

$$\mu^* = \inf_{n\geq N} \min_{T=k,k-k_i,i\in I_m} \{\frac{1}{p}\mu^*(1+\mu^*)^k + \frac{1}{p}\sum_{i=1}^{m}(\frac{1}{T}\sum_{j=n}^{n+T-1} q_{j+k}^i)(1+\mu^*)^{k-k_i}\},$$

hence

$$\inf_{n\geq N,\mu>0} \min_{T=k,k-k_i,i\in I_m} \{\frac{1}{p}(1+\mu)^k + \frac{1}{p\mu}\sum_{i=1}^{m}(\frac{1}{T}\sum_{j=n}^{n+T-1} q_{j+k}^i)(1+\mu)^{k-k_i}\} \leq 1,$$

a contradiction. Therefore $\lim_{k\to\infty} \mu_k = +\infty$. Hence from (2.12) and (2.13) we have

$$\frac{1}{T}\sum_{j=n}^{n+T-1} w_j \to \infty \text{ as } n \to \infty$$

and hence from (2.8)

$$\frac{z_{n+T}}{z_n} \to \infty \text{ as } n \to \infty. \tag{2.14}$$

On the other hand, $z_n > -py_{n-k}$, therefore from Lemma 2.1

$$\Delta z_n = -\sum_{i=1}^{m} q_n^i y_{n-k_i} < \frac{1}{p}\sum_{i=1}^{m} q_n^i z_{n+k-k_i} < (\frac{1}{p}\sum_{i=1}^{m} q_n^i) z_{n+k-\bar{k}}.$$

Hence

$$\sum_{j=n}^{n+[\frac{k-\bar{k}}{2}]-1} (\frac{1}{p}\sum_{i=1}^{m} q_j^i)\frac{z_{n+k-\bar{k}}}{z_{n+[\frac{k-\bar{k}}{2}]}} \leq 1. \tag{2.15}$$

By (2.10) and (2.15) we have

$$z_{n+k-\bar{k}}/z_{n+[\frac{k-\bar{k}}{2}]} \text{ is bounded,}$$

which contradicts (2.14). The proof is completed.

For $m = 1$, from (2.9) the following theorem holds.

Theorem 2.2. *Assume that*

(1) $p > 1, k \geq k_1 + 2$;

(2) $q_n > 0$ *and*

$$\liminf_{n\to\infty} \sum_{j=n}^{n+[\frac{k-k_1}{2}]-1} q_j > 0;$$

(3)

$$\inf_{n\geq N,\mu>0}\min \{\frac{(1+\mu)^k}{p(k-k_1)}\sum_{i=n}^{n+k-k_1-1}\frac{q_i}{q_{i+k}}+\frac{(1+\mu)^{k-k_1}}{p\mu(k-k_1)}\sum_{i=n}^{n+k-k_1-1}q_i\}>1. \tag{2.16}$$

Then every solution of (1.2) *is oscillatory.*

Theorem 2.3. *Assume that*

(1) $p>1, k>k_i, i\in I_m$;

(2)

$$\liminf_{n\to\infty}\,(\sum_{i=1}^{m}q_n^i)>0;$$

(3) *there exist* $\mu>0$ *and* N *such that*

$$\sup_{n\geq N,\ T=k,k-k_i,i\in I_m}[\frac{1}{p}(1+\mu)^k+\frac{1}{p\mu}\sum_{i=1}^{m}\frac{1}{T}(\sum_{j=n}^{n+T-1}q_{j+k}^i)(1+\mu)^{k-k_i}]\leq 1. \tag{2.17}$$

Then equation (1.2) *has a positive solution.*

Proof. Define

$$\{\lambda_n^{(l)}\}=\{0\}, n=N,N+1,\dots,$$

$$\lambda_n^{(l+1)}=\frac{1}{p}\lambda_{n+k}^{(l)}\prod_{i=n}^{n+k-1}(1+\lambda_i^{(l)})+\frac{1}{p}\sum_{i=1}^{m}q_{n+k}^i\prod_{i=n}^{n+k-k_i-1}(1+\lambda_i^{(l)})$$

$$n=N,N+1,\dots,l=1,2,\dots$$

Clearly $\lambda_n^{(l)}\leq\lambda_n^{(l+1)}, l=1,2,\dots,n=N,N+1,\dots$. We claim that

$$\frac{1}{T}\sum_{i=n}^{n+T-1}\lambda_i^{(l)}\leq\mu,\ n\geq N. \tag{2.18}$$

In fact, $l=1$ is true. Assume that (2.18) is true for some l', then

$$\begin{aligned}\frac{1}{T}\sum_{i=n}^{n+T-1}\lambda_i^{(l'+1)}=&\frac{1}{T}\sum_{i=n}^{n+T-1}\{\frac{1}{p}\lambda_{i+k}^{(l')}\prod_{j=i}^{i+k-1}(1+\lambda_j^{(l')})+\frac{1}{p}\sum_{j=1}^{m}q_{i+k}^j\prod_{j=i}^{i+k-k_j-1}(1+\lambda_j^{(l')})\}\\ \leq&\frac{\mu}{p}(1+\mu)^k+\frac{1}{pT}\sum_{i=n}^{n+T-1}(\sum_{j=1}^{m}q_{i+k}^j)(1+\mu)^{k-k_i}\leq\mu.\end{aligned}$$

Hence $\{\lambda_n^{(l)}\}\to\{\lambda_n\}$ as $l\to\infty$, $n=N,N+1,\dots$, and

$$\frac{1}{T}\sum_{i=n}^{n+T-1}\lambda_i\leq\mu,\ n\geq N$$

and

$$\lambda_n = \frac{1}{p}\lambda_{n+k} \prod_{i=n}^{n+k-1} (1+\lambda_i) + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \prod_{i=n}^{n+k-k_i-1} (1+\lambda_i), \ n \geq N. \tag{2.19}$$

Set $z_N = 1$, $\frac{z_{n+1}}{z_n} = 1 + \lambda_n$, $n = N, N+1, \ldots$, therefore

$$\frac{z_n}{z_N} = z_n = \prod_{i=N}^{n-1}(1+\lambda_n), \ \lambda_n = \frac{\Delta z_n}{z_n},$$

hence (2.19) becomes

$$\frac{\Delta z_n}{z_N} = \frac{1}{p}\frac{\Delta z_{n+k}}{z_{n+k}}\frac{z_{n+k}}{z_n} + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \frac{z_{n+k-k_i}}{z_n},$$

and hence

$$\Delta z_n = \frac{1}{p}\Delta z_{n+k} + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i z_{n+k-k_i}$$

$$p\Delta z_n = \Delta z_{n+k} + \sum_{i=1}^{m} q_{n+k}^i z_{n+k-k_i},$$

which implies that equation (1.2) has a positive solution

$$z_N = 1, \ z_n = \prod_{i=N}^{n-1}(1+\lambda_i), \ n = N+1, \ldots$$

For $m = 1$, we have the following

Theorem 2.4. *Assume that*
(1) $p > 1, k > k_1$;
(2) $\liminf_{n\to\infty} q_n > 0$;
(3) *There exist* $\mu > 0$ *and* N *such that*

$$\sup_{n\geq N}\{\frac{q_n}{pq_{n+k}}(1+\mu)^k + \frac{q_n}{p\mu}(1+\mu)^{k-k_1}\} \leq 1. \tag{2.20}$$

Then equation (1.2) has a positive solution.

Corollary 2.1. *If* $m = 1, p > 1, k > k_1, q_n = q > 0$, *then every solution of (1.2) is oscillatory if and only if*

$$\inf_{n\geq N,\mu>0}\{\frac{1}{p}(1+\mu)^k + \frac{q}{p\mu}(1+\mu)^{k-k_i}\} > 1. \tag{2.21}$$

Remark 2.2. Theorem 2.2 includes Theorem 4.1 of [8] and Theorem 4 (a) in [4], since

$$\min_{\mu>0} \frac{(1+\mu)^{k-k_1}}{\mu} = \frac{(k-k_1)^{k-k_1}}{(k-k_1-1)^{k-k_1-1}}.$$

Remark 2.3. Equation (1.2) with constant coefficients has been studied in [3]. The result in Corollary 2.1 is better.

As an application of Theorem 2.1 and 2.3 we consider equation (1.2) with periodic coefficients, i.e., we assume that there exists an integer $\delta > 0$ such that

$$q^i_{n+\delta} = q^i_n, \ i \in I_m \text{ for any } n. \tag{2.22}$$

Then

$$\frac{1}{\delta} \sum_{j=n}^{n+\delta-1} q^i_j = \bar{q}^i \tag{2.23}$$

is a constant, $i \in I_m$.

Theorem 2.5. *Assume that $p \geq 1$ and there exist positive integers m_0 and $m_i, i \in I_m$ such that $k = m_0\delta$, $k_i = m_i\delta$, $k - k_i \geq 2$, $i \in I_m$. Then following statements are equivalent.*

(1) *Every solution of* (1.2) *is oscillatory;*

(2) *Every solution of the neutral difference equation with constant coefficients*

$$\Delta(y_n - py_{n-k}) + \sum_{i=1}^{m} \bar{q}^i y_{n-k_i} = 0 \tag{2.24}$$

is oscillatory.

Proof. Suppose that (2.24) has a nonoscillatory solution. Then the characteristic equation [6]

$$(\lambda - 1)(1 - p\lambda^{-k}) + \sum_{i=1}^{m} \bar{q}^i \lambda^{-k_i} = 0 \tag{2.25}$$

has a positive root. It is easy to see that $\lambda > 1$. Set $\lambda = 1 + \mu$. Then (2.25) is reduced to

$$\frac{1}{p}(1+\mu)^k + \frac{1}{p\mu} \sum_{i=1}^{m} \bar{q}^i (1+\mu)^{k-k_i} = 1, \tag{2.26}$$

which implies that Theorem 2.3 holds. Hence (1.2) has a positive solution, a contraction. If (2) holds, then (2.25) has no real roots. Therefore

$$(\lambda - 1)(1 - p\lambda^{-k}) + \sum_{i=1}^{m} \bar{q}^i \lambda^{-k_i} > 0. \tag{2.27}$$

Setting $\lambda = 1 + \mu$, then we have

$$\frac{1}{p}(1+\mu)^k + \frac{1}{p\mu}\sum_{i=1}^{m} \tilde{q}^i (1+\mu)^{k-k_i} > 1. \tag{2.28}$$

By Theorem 2.1, every solution of (1.2) is oscillatory. The proof is completed.

Remark 2.4. Using the average method to Theorems 3.5 and 3.6 in [8], it is not difficult to show that Theorem 2.5 is true for $p \in [0,1]$ too, where $k > \tilde{k}$ is not required.

Now we present a comparison result for oscillation of (1.2). we consider (1.2) associated with

$$\Delta(y_n - Py_{n-k}) + \sum_{i=1}^{m} Q_n^i y_{n-k_i} = 0. \tag{2.29}$$

Theorem 2.6. *Assume that*

(1) $k \geq \tilde{k} + 2$;

(2) $P > 1, Q_{n-k}^i \geq Q_n^i > 0, \sum\limits_{j=N}^{\infty} \sum\limits_{i=1}^{n} Q_j^i = \infty$ *and*

$$\frac{P}{p} \leq \frac{Q_n^i}{q_n^i} \leq 1, \;\; n = N, N+1, \dots, \;\; i \in I_m. \tag{2.30}$$

Then if every solution of (1.2) *is oscillatory, so is* (2.29).

Proof. Suppose not. Let $\{x_n\}$ be a positive solution of (2.29). By Lemma 2.2,

$$u_n \geq \frac{1}{P} u_{n+k} \prod_{i=n}^{n+k-1} (1+u_i) + \frac{1}{P}\sum_{i=1}^{m} Q_{n+k}^i \prod_{i=n}^{n+k-k_i-1} (1+u_i)$$

where

$$u_i = \frac{\Delta(x_i - px_{i-k})}{x_i - px_{i-k}} > 0.$$

In view of (2)

$$u_n \geq \frac{1}{p} u_{n+k} \prod_{i=n}^{n+k-1} (1+u_i) + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \prod_{i=n}^{n+k-k_i-1} (1+u_i). \tag{2.31}$$

Define

$$\{\lambda_n^{(l)}\} = \{u_n\}, \;\; n = N, N+1, \dots$$

and

$$\lambda_n^{(l+1)} = \frac{1}{p}\lambda_{n+k}^{(l)} \prod_{i=n}^{n+k-1} (1+\lambda_i^{(l)}) + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \prod_{i=n}^{n+k-k_i-1} (1+\lambda_i^{(l)}).$$

$$n = N, N+1, \ldots, \; l = 1, 2, \ldots \tag{2.32}$$

In view of (2.31), we have

$$\lambda_n^{(l+1)} \le \lambda_n^{(l)}, \; l = 1, 2, \ldots, \; n = N, N+1, \ldots$$

Hence

$$\lim_{l \to \infty} \lambda_n^{(l)} = \lambda_n$$

exists and $\lambda_n > 0$ satisfies

$$\lambda_n = \frac{1}{p}\lambda_{n+k} \prod_{i=n}^{n+k-1} (1+\lambda_i) + \frac{1}{p}\sum_{i=1}^{m} q_{n+k}^i \prod_{i=n}^{n+k-k_i-1} (1+\lambda_i).$$

Similar to Theorem 2.3, we obtain a positive solution of (1.2)

$$z_N = 1, \; z_n = \prod_{i=N}^{n-1} (1+\lambda_i), \; n = N+1, \ldots$$

which contradicts the assumption. The proof is completed.

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