TAMKANG JOURNAL OF MATHEMATICS
Volume 25, Number 4, Winter 1994

ASYMPTOTIC NONNULL DISTRIBUTION OF LRC FOR TESTING $H: \mu = \mu_0; \Sigma = \sigma^2 I_p$ IN GAUSSIAN POPULATION

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Abstract. In this paper asymptotic expansions of the nonnull distribution of the likelihood ratio statistic for testing $H : \mu = \mu_0; \Sigma = \sigma^2 I_p$, against alternatives which are close to H, for Gaussian population, have been derived.

1. Introduction

Let $x_1, \ldots, x_N$ be a random sample of size N from a p-variate normal population with unknown mean vector μ and positive definite covariance matrix Σ. The likelihood ratio criterion for testing $H : \mu = \mu_0; \Sigma = \sigma^2 I_p$, where μ_0 is a known p-vector, I_p is the identity matrix of order p, $\sigma^2 > 0$ is an unknown scalar; can be expressed as

$$\Lambda = [p^p |A| / \{\operatorname{tr} A + N(\bar{x} - \mu_0)'(\bar{x} - u_0)\}^p]^{N/2}, N - 1 = n \geq p,$$

where $\bar{x} = N^{-1} \sum_{j=1}^N x_j$ and $A = \sum_{j=1}^N (x_j - x)(x_j - x)'$.

Khatri and Srivastava (1973) pointed out that the test based on Λ is unbiased and have also derived the nonnull distribution of Λ in multiple series of zonal polynomials and G-functions. Singh (1980) derived the null distribution of Λ and also computed the percentage points. Nagar and Gupta (1986) derived the nonnull distribution in series involving H-functions.

In this paper we derive the asymptotic nonnull distribution of a multiple of $-2 \log \Lambda$ for certain alternatives. First in Section 2, some preliminary results are given. The nonnull moments of Λ are derived in Section 3. This moment expression involves zonal polynomials and some coefficients, β_j, which are given explicitly. In the end asymptotic nonnull distribution of $-2 \log \Lambda$ is given in a series of central/non-central chi-squared distributions.

Received August 2, 1993.

1991 *Mathematics Subject Classification.* Primary 62H10; Secondary 62H15.

Key words and phrases. Distribution, testing, moments, characteristic function, zonal polynomials, asymptotics.

2. Preliminaries

Let $C_K(B)$ denote the zonal polynomial, a symmetric function in the roots of a symmetric matrix B (James, 1964) of degree k corresponding to the partition $K = (k_1, \dots, k_p), k_1 \geq \dots \geq k_p \geq 0, \sum_{i=1}^{p} k_i = k$ and

$$(a)_K = \prod_{j=1}^{p} \Big(a - \frac{j-1}{2}\Big)_{k_j}, \quad (a)_k = \prod_{j=1}^{k} (a + j - 1), \quad (a)_0 = 1.$$

If a is such that all the gamma functions are defined then $(a)_K = \frac{\Gamma_p(a,K)}{\Gamma_p(a)}$, where

$$\Gamma_p(a, K) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\Big(a - \frac{j-1}{2} + k_j\Big)$$
$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\Big(a - \frac{j-1}{2}\Big).$$

We now state the following result available in Khatri and Srivastava (1974).

Lemma 2.1. *Let $||B||$ = maximum characteristic root of $B \leq 1$ and $d_i = \beta\, tr\{B^i(I-B)^{-i}\}, i = 1, 2, 3,$ and $4, \beta > 0$. Then*

(a) $\sum_{k=0}^{\infty} \sum_{K} k(\beta)_K \frac{C_K(B)}{k!} = d_1 |I - B|^{-\beta}$

(b) $\sum_{k=0}^{\infty} \sum_{K} k(k-1)(\beta)_K \frac{C_K(B)}{k!} = (d_2 + d_1^2)|I - B|^{-\beta}$

(c) $\sum_{k=0}^{\infty} \sum_{K} k(k-1)(k-2)(\beta)_K \frac{C_K(B)}{k!} = (2d_3 + 3d_2 d_1 + d_1^3)|I - B|^{-\beta}$

(d) $\sum_{k=0}^{\infty} \sum_{K} k(k-1)(k-2)(k-3)(\beta)_K \frac{C_K(B)}{k!} = (6d_4 + 8d_1 d_2 + 3d_2^2 + 6d_1^2 d_2 + d_1^4)|I - B|^{-\beta}$

Lemma 2.2. *Let the asymptotic series $\sum_{j=1}^{\infty} \alpha_j y^j$ converge to the function $g(y)$ in the neighbourhood of $y = 0$. Then we have (Kalinin and Shalaevskii, 1971)*

$$\exp[g(y)] = \sum_{j=0}^{\infty} \beta_j y^j$$

where $\beta_0 = 1$, and

$$\beta_j = \frac{1}{j} \sum_{\ell=1}^{j} \ell \alpha_\ell \beta_{j-\ell}, j = 1, 2, \dots.$$

Also differentiating the identity

$$\exp\left(\sum_{j=1}^{\infty} \alpha_j y^j\right) = \sum_{j=0}^{\infty} \beta_j y^j$$

with respect to y *once, twice, ..., we get the following identities.*

(i)
$$\sum_{j=1}^{\infty} j\beta_j y^j = \exp\left[\sum_{j=1}^{\infty} \alpha_j y^j\right]\left[\sum_{j=1}^{\infty} j\alpha_j y^j\right]$$

(ii)
$$\sum_{j=2}^{\infty} j(j-1)\beta_j y^j = \exp\left[\sum_{j=1}^{\infty} \alpha_j y^j\right]\left[\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)^2 + \sum_{j=2}^{\infty} j(j-1)\alpha_j y^j\right]$$

(iii)
$$\sum_{j=3}^{\infty} j(j-1)(j-2)\beta_j y^j = \exp\left[\sum_{j=1}^{\infty} \alpha_j y^j\right]\left[\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)^3 \right.$$
$$\left. +3\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)\left(\sum_{j=2}^{\infty} j(j-1)\alpha_j y^j\right) + \sum_{j=3}^{\infty} j(j-1)(j-2)\alpha_j y^j\right]$$

(iv)
$$\sum_{j=4}^{\infty} j(j-1)(j-2)(j-3)\beta_j y^j = \exp\left[\sum_{j=1}^{\infty} \alpha_j y^j\right]\left[\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)^4 \right.$$
$$+ 6\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)^2\left(\sum_{j=2}^{\infty} j(j-1)\alpha_j y^j\right) + 4\left(\sum_{j=1}^{\infty} j\alpha_j y^j\right)$$
$$\cdot\left(\sum_{j=3}^{\infty} j(j-1)(j-2)\alpha_j y^j\right) + 3\left(\sum_{j=2}^{\infty} j(j-1)\alpha_j y^j\right)^2$$
$$\left. + \sum_{j=4}^{\infty} j(j-1)(j-2)(j=3)\alpha_j y^j\right].$$

3. Nonnull Moments of Λ

The nonnull moments of Λ are available in Khatri and Srivastava (1973). But certain coefficients, involved in the expression which are necessary for deriving the asymptotic distribution, have not been cited explicitly in that paper. Nagar and Gupta (1986) have also derived the nonnull moments in series of Lauricella's hypergeometric functions. Using Equation (2.3) of Nagar and Gupta (1986), the h^{th} moment of Λ is obtained as

$$E(\Lambda^h) = \frac{p^{Nph/2}|\Sigma^{-1}|^{N/2}\Gamma_p\left(\frac{Nh}{2}+\frac{N-1}{2}\right)\exp\left[-\frac{1}{2}N(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\right]}{\Gamma(\mathrm{ph})\Gamma_p\left(\frac{N-1}{2}\right)}$$
$$\int_0^\infty x^{\mathrm{hp}-1}\cdot\exp\left[\frac{1}{2}N(\mu-\mu_0)'\Sigma^{-1}(\Sigma^{-1}+xI)^{-1}\Sigma^{-1}(\mu-\mu_0)\right]|\Sigma^{-1}+xI|^{-N(1+h)/2}dx,$$
$$\mathrm{Re}(h) > -(N-p)/2. \tag{3.1}$$

Expanding $|\Sigma^{-1}+xI|^{-N(1+h)/2}$ in terms of zonal polynomials,

$$|\Sigma^{-1}+xI|^{-N(1+h)/2} = (1+qx)^{-pN(1+h)/2}q^{pN(1+h)/2} \cdot\sum_{k=0}^\infty\sum_K \frac{(N(1+h)/2)_K C_K(I-q\Sigma^{-1})}{(1+qx)^k k!} \tag{3.2}$$

where $0 < q < \infty$ and can be so chosen that the expansion in the series form is valid, i.e. $||I-q\Sigma^{-1}|| < 1+qx$. One can choose q such that $0 < q < 2\lambda_p$ where λ_p is the largest characteristic root of Σ (Khatri and Srivastava, 1973). Also, note that

$$\begin{aligned}&\frac{1}{2}N(\mu-\mu_0)'\Sigma^{-1}(\Sigma^{-1}+xI)^{-1}\Sigma^{-1}(\mu-\mu_0)\\ =&\frac{1}{2}Nq(1+qx)^{-1}(\mu-\mu_0)'\Sigma^{-1}(I-(1+qx)^{-1}(I-q\Sigma^{-1}))^{-1}\Sigma^{-1}(\mu-\mu_0)\\ =&\sum_{j=1}^\infty \alpha_j(1+qx)^{-j}, \quad ||I-q\Sigma^{-1}||<1\end{aligned} \tag{3.3}$$

where

$$\alpha_j = \frac{1}{2}Nq(\mu-\mu_0)'\Sigma^{-1}(I-q\Sigma^{-1})^{j-1}\Sigma^{-1}(\mu-\mu_0). \tag{3.4}$$

So that using Lemma 2.2, we have

$$\begin{aligned}\exp\left[\frac{1}{2}N(\mu-\mu_0)'\Sigma^{-1}(\Sigma^{-1}+xI)^{-1}\Sigma^{-1}(\mu-\mu_0)\right] &= \exp\left[\sum_{j=1}^\infty \alpha_j(1+xq)^{-j}\right]\\ &= \sum_{j=0}^\infty \beta_j(1+qx)^{-j}\end{aligned} \tag{3.5}$$

where $\beta_0 = 1$, and

$$\beta_j = \frac{1}{j}\sum_{\ell=1}^{j} \ell\alpha_\ell\beta_{j-\ell}, \quad j = 1, 2, \ldots. \tag{3.6}$$

Substituting (3.2) and (3.5) in (3.1) and integrating out term by term, we get the h^{th} nonnull moment of Λ as

$$E(\Lambda^h) = \frac{p^{Nph/2}|q\Sigma^{-1}|^{N/2}\Gamma_p\Big(N(1+h)/2 - \frac{1}{2}\Big)\exp\Big[-\frac{1}{2}N(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\Big]}{\Gamma_p((N-1)/2)} \cdot \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\beta_j\sum_K (N(1+h)/2)_K \frac{C_K(I-q\Sigma^{-1})}{k!}\frac{\Gamma[N_p/2+j+k]}{\Gamma[N_p(1+h)/2+j+k]}. \tag{3.7}$$

4. Asymptotic Nonnull Distribution of Λ

In this section we derive the asymptotic nonnull distribution of a multiple of $-2\ell n\Lambda$ for certain local alternatives (see Khatri and Srivastava, 1974). Let

$$\alpha = \frac{p^2(2p^2+9p+11)-4}{12p(p^2+3p-2)} \tag{4.1}$$

$$\rho = 1 - 2\frac{\alpha}{N}, \quad M = \rho N \tag{4.2}$$

and let us consider the alternative hypothesis

$$A_N : I - q\Sigma^{-1} = \frac{2}{M}P; \quad q^{-1/2}(\mu-\mu_0) = \Big(\frac{2}{M}\Big)^{1/2}\delta \tag{4.3}$$

where P is a fixed matrix as $M \to \infty$. Then the characteristic function of $-2p\ell n\Lambda$ is derived from (3.7) as

$$\phi(t) = \Big|I - \frac{2}{M}P\Big|^{M/2+\alpha}\exp\Big[-(\alpha+M/2)\frac{2}{M}\delta'\Big(I-\frac{2P}{M}\Big)\delta\Big]\sum_{j=0}^{\infty}\beta_j\phi_j(t) \tag{4.4}$$

where β_j is given by the recurrence relation (3.6) and the coefficient α_j is given by

$$\alpha_j = \Big(\frac{2}{M}\Big)^{j-1}\Big[\sigma_{j-1} + \frac{2}{M}(\alpha\sigma_{j-1} - 2\sigma_j) + \frac{4}{M^2}(\sigma_{j+1} - 2\alpha\sigma_j) + \frac{8}{M^3}\alpha\sigma_{j+1}\Big] \tag{4.5}$$

with $\sigma_j = \delta' P^j\delta$, $j = 1, 2, \ldots$, $\sigma_0 = \delta'\delta$,

$$\phi_j(t) = \sum_{k=0}^{\infty}\sum_K (\alpha + Mg/2)_K \frac{C_K\Big(\frac{2P}{M}\Big)}{k!}\phi_{k,j}(t) \tag{4.6}$$

$g = 1 - 2\omega t$, $\omega = (-1)^{1/2}$. Here $\phi_{k,j}(t)$ is the characteristic function of $-2p\ell n W_{kj}$, where the random variable W_{kj} has the h^{th} moment as

$$E(W_{kj}^h) = p^{\text{ph}N/2} \frac{\Gamma[Np/2 + j + k]}{\Gamma[Np(1+h)/2 + j + k]} \prod_{i=1}^{p} \Big\{ \frac{\Gamma[N(1+h)/2 - i/2]}{\Gamma[(N-i)/2]} \Big\}.$$

Using Equation (16) of Anderson (1984, p. 314),

$$\begin{aligned}\phi_{k,j}(t) =& g^{-f/2}\Big[1 - \frac{1}{M_p}\{j(j-1) + k(k-1) + 2\alpha pj + 2\alpha pk + 2kj\}(g^{-1} - 1) \\ &+ \frac{1}{2(Mp)^2}\{k(k-1)(k-2)(k-3) + 4(1 + j + \alpha p)k(k-1)(k-2) \\ &+ 2(1 + 3j^2 + 6\alpha pj + 2\alpha^2p^2 + 4\alpha p + 3j)k(k-1) \\ &+ 4(\alpha^2p^2 + \alpha pj + 2\alpha^2p^2j + 3\alpha pj^2 + j^3)k + j^2(j - 1 + 2\alpha p)^2\}(g^{-1} - 1)^2 \\ &+ \frac{\gamma}{(Mp)^2}(g^{-2} - 1) + \frac{2}{3(Mp)^2}\Big\{j^3 + \Big(3\alpha p - \frac{3}{2}\Big)j^2 + \Big(3\alpha^2p^2 - 3\alpha p + \frac{1}{2}\Big)j \\ &+ k(k-1)(k-2) + 3\Big(\alpha p + j + \frac{1}{2}\Big)k(k-1) + 3(j^2 + 2\alpha pj + \alpha^2p^2)k\Big\} \\ &\cdot (g^{-2} - 1) + O(M^{-3})\Big] \qquad (4.7)\end{aligned}$$

where $f = v + 2(k + j)$, $v = \frac{1}{2}(p^2 + 3p - 2)$, and

$$\gamma = \frac{1}{48}p^3(p+1)(p+2)(p+3) - \frac{1}{2}\alpha^2p^2(p^2 + 3p - 2). \qquad (4.8)$$

Substituting (4.7) in (4.6) and using Lemma 2.1, the approximate value of $\phi_j(t)$ is

$$\begin{aligned}\phi_j(t) =& g^{-v/2-j}\Big|I - \frac{2P}{Mg}\Big|^{-Mg/2-\alpha}\Big[1 - \frac{1}{Mp}\{j(j-1) + 2(\alpha p + s_1)j + s_1^2 + 2\alpha p s_1\}(g^{-1} - 1) \\ &- \frac{2}{M^2pg}\{2s_1(\alpha s_1 + s_2) + s_2 + (2\alpha p + 2j)(\alpha s_1 + s_2)\}(g^{-1} - 1) \\ &+ \frac{1}{2(Mp)^2}\{j(j-1)(j-2)(j-3) + 4(1 + \alpha p + s_1)j(j-1)(j-2) \\ &+ (2 + 8\alpha p + 12s_1 + 6s_1^2 + 12\alpha p s_1 + 4\alpha^2p^2)j(j-1) \\ &+ (4s_1 + 12s_1^2 + 16\alpha p s_1 + 4\alpha^2p^2 + 4s_1^3 + 12\alpha p s_1^2 + 8\alpha^2p^2s_1)j + s_1^4 \\ &+ 4(1 + \alpha p)s_1^3 + 2(1 + 2\alpha^2p^2 + 4\alpha p)s_1^2 + 4\alpha^2p^2s_1\}(g^{-1} - 1)^2 + \frac{\gamma}{(Mp)^2}(g^{-2} - 1) \\ &+ \frac{2}{3(Mp)^2}\{j(j-1)(j-2) + 3\Big(\alpha p + s_1 + \frac{1}{2}\Big)j(j-1) + 3(s_1 + \alpha^2p^2 + 2\alpha p s_1 + s_1^2)j \\ &+ s_1^3 + 3\Big(\alpha p + \frac{1}{2}\Big)s_1^2 + 3\alpha^2p^2s_1\}(g^{-2} - 1) + O(M^{-3})\Big] \qquad (4.9)\end{aligned}$$

where $s_i = \text{tr}P^i$, $i = 1, 2$ and 3. Substituting (4.9) in (4.4) and using equalities (i)-(iv) of Lemma 2.2 the approximate value of $\phi(t)$ is obtained as

$$\phi(t) = \left|I - \frac{2}{M}P\right|^{M/2+\alpha} \left|I - \frac{2}{Mg}P\right|^{-Mg/2-\alpha} g^{-v/2} \exp\Big[-(\alpha + M/2)\frac{2}{M}\delta'\Big(I - \frac{2P}{M}\Big)\delta + \sum_{j=1}^{\infty} \alpha_j g^{-j}\Big]\Big[1 + \frac{1}{M}\Big\{\sum_{i=0}^{3} T_i g^{-i}\Big\} + \frac{1}{M^2}\Big\{\sum_{i=0}^{6} B_i g^{-i}\Big\} + O(M^{-3})\Big] \tag{4.10}$$

where

$$T_0 = \frac{1}{p}(s_1^2 + 2\alpha p s_1), \quad T_1 = -T_0 + \frac{2\sigma_0}{p}(s_1 + \alpha p)$$

$$T_2 = \frac{1}{p}\sigma_0^2 - \frac{2\sigma_0}{p}(s_1 + \alpha p), \quad T_3 = -\frac{1}{p}\sigma_0^2$$

$$B_0 = \frac{1}{p^2}\Big[-\gamma + \frac{1}{2}s_1^4 + \Big(\frac{4}{3} + 2\alpha p\Big)s_1^3 + 2\alpha p(1 + \alpha p)s_1^2\Big]$$

$$B_1 = \frac{4}{p}(s_1 + \alpha p)(\alpha\sigma_0 - 2\sigma_1) + \frac{\sigma_0}{p^2}\{4s_1^2 + 4\alpha p s_1 + 2s_1^3 + 6\alpha p s_1^2 + 4\alpha^2 p^2 s_1\} - \frac{1}{p^2}s_1^4 - 4(1 + \alpha p)\frac{s_1^3}{p^2} + \frac{2}{p}(1 + 2s_1 + 2\alpha p)s_2 - (2 + 4\alpha p + 4\alpha^2 p^2)\frac{s_1^2}{p^2}$$

$$B_2 = -\frac{4}{p}(\alpha\sigma_0 - 4\sigma_1)(s_1 + \alpha p) + \frac{4}{p}\{\sigma_0(\alpha\sigma_0 - 2\sigma_1) + \sigma_1\} + \frac{\sigma_0}{p^2}\{2\alpha p + 4s_1 + 3s_1^2 + 6\alpha p s_1 + 2\alpha^2 p^2\} - \frac{\sigma_0}{p^2}\{4s_1 + 12s_1^2 + 12\alpha p s_1 + 4\alpha^2 p^2 + 4s_1^3 + 12\alpha p s_1^2 + 8\alpha^2 p^2 s_1 - 4p s_2\} + \frac{1}{p^2}\Big\{\frac{1}{2}s_1^4 + 2\Big(\frac{4}{3} + \alpha p\Big)s_1^3 + 2(1 + \alpha p + \alpha^2 p^2)s_1^2 - 2p(2s_1 + 2\alpha p + 1)s_2\Big\} + \frac{\gamma}{p^2}$$

$$B_3 = \frac{-8\sigma_1}{p}(s_1 + \alpha p) - \frac{4}{p}(\alpha\sigma_0^2 - 4\sigma_0\sigma_1 + \sigma_1) + \frac{1}{p^2}\{2(1 + \alpha p + s_1)\sigma_0^3 - \sigma_0^2(2 + 8\alpha p + 12s_1 + 6s_1^2 + 12\alpha p s_1 + 4\alpha^2 p^2) + \sigma_0(4s_1 + 8s_1^2 + 8\alpha p s_1 + 4\alpha^2 p^2 + 2s_1^3 + 6\alpha p s_1^2 + 4\alpha^2 p^2 s_1 - 4p s_2)\} - \frac{2}{3p^2}\sigma_0^3$$

$$B_4 = -\frac{8}{p}\sigma_0\sigma_1 + \frac{1}{p^2}\Big\{\frac{1}{2}\sigma_0^4 - 4\sigma_0^3(1 + \alpha p + s_1) + \sigma_0^2(2 + 6\alpha p + 8s_1 + 3s_1^2 + 6\alpha p s_1 + 2\alpha^2 p^2)\Big\}$$

$$B_5 = \frac{1}{p^2}\Big\{-\sigma_0^4 + 2(1 + \alpha p + s_1)\sigma_0^3 + \frac{2}{3}\sigma_0^3\Big\}, \quad \text{and} \quad B_6 = \frac{1}{2p^2}\sigma_0^4.$$

Now using the asymptotic expansion

$$
\begin{aligned}
&\left|I-\frac{2P}{M}\right|^{\alpha+M/2}\left|I-\frac{2P}{Mg}\right|^{-\alpha-Mg/2}\exp\left[-(\alpha+M/2)\frac{2}{M}\delta'\left(I-\frac{2P}{M}\right)\delta\right]\\
&\times\exp\left[\sum_{j=1}^{\infty}\left(\frac{2}{M}\right)^{j-1}\left\{\sigma_{j-1}+\frac{2}{M}(\alpha\sigma_{j-1}-2\sigma_j)+\frac{4}{M^2}(\sigma_{j+1}-2\alpha\sigma_j)\frac{8\alpha\sigma_{j+1}}{M^3}\right\}g^{-j}\right]\\
&=\exp\Bigg[-\sigma_0+\frac{2}{M}(\sigma_1-\alpha\sigma_0)+\frac{4}{M^2}\alpha\sigma_1+\sum_{j=1}^{\infty}\left(\frac{2}{M}\right)^{j-1}\left\{\sigma_{j-1}+\frac{2}{M}(\alpha\sigma_{j-1}-2\sigma_j)\right.\\
&\quad\left.+\frac{4}{M^2}(\sigma_{j+1}-2\alpha\sigma_j)\frac{8}{M^3}\alpha\sigma_{j+1}\right\}g^{-j}+\left(\alpha+\frac{M}{2}\right)\ell n\left|I-\frac{2P}{M}\right|-\left(\alpha+\frac{Mg}{2}\right)\ell n\left|I-\frac{2P}{Mg}\right|\Bigg]\\
&=\exp\Bigg[-\sigma_0+\sigma_0g^{-1}+\frac{2}{M}\left\{\sigma_1-\alpha\sigma_0-\alpha s_1-\frac{s_2}{2}+\left(\alpha\sigma_0-2\sigma_1+\alpha s_1+\frac{s_2}{2}\right)g^{-1}+\sigma_1g^{-2}\right\}\\
&\quad+\frac{4}{M^2}\left\{\alpha\sigma_1-\frac{\alpha s_2}{2}-\frac{s_3}{3}+(\sigma_2-2\alpha\sigma_1)g^{-1}+\left(\alpha\sigma_1-2\sigma_2+\frac{\alpha s_2}{2}+\frac{s_3}{3}\right)g^{-2}+\sigma_2g^{-3}\right\}\\
&\quad+O(M^{-3})\Bigg]\\
&=\exp[\sigma_0(g^{-1}-1)]\Bigg[1+\frac{2}{M}\left\{\sigma_1-\alpha\sigma_0-\alpha s_1-\frac{s_2}{2}+\left(\alpha\sigma_0-2\sigma_1+\alpha s_1+\frac{s_2}{2}\right)g^{-1}+\sigma_1g^{-2}\right\}\\
&\quad+\frac{4}{M^2}\left\{\alpha\sigma_1-\frac{\alpha s_2}{2}-\frac{s_3}{3}+\frac{1}{2}\left(\sigma_1-\alpha\sigma_0-\alpha s_1-\frac{s_2}{2}\right)^2+(\sigma_2-2\alpha\sigma_1)g^{-1}\right.\\
&\quad+\left(\sigma_1-\alpha\sigma_0-\alpha s_1-\frac{s_2}{2}\right)\left(\alpha\sigma_0-2\sigma_1+\alpha s_1-\frac{s_2}{2}\right)g^{-1}+\left(\alpha\sigma_1-2\sigma_2+\frac{\alpha s_2}{2}+\frac{s_3}{3}\right)g^{-2}\\
&\quad+\frac{1}{2}\left(\alpha\sigma_0-2\sigma_1+\alpha s_1+\frac{s_2}{2}\right)^2g^{-2}+\sigma_1\left(\sigma_1-\alpha\sigma_0-\alpha s_1-\frac{s_2}{2}\right)g^{-2}+\sigma_2g^{-3}\\
&\quad\left.+\left(\alpha\sigma_0-2\sigma_1+\alpha s_1+\frac{s_2}{2}\right)\sigma_1g^{-3}+\frac{1}{2}\sigma_1^2g^{-4}\right\}+p(M^{-3})\Bigg] \qquad (4.11)
\end{aligned}
$$

in (4.10), the final asymptotic expansion of the characteristic function $\phi(t)$ is

$$\phi(t)=\exp[\sigma_0(g^{-1}-1)]g^{-v/2}\left[1+\frac{1}{M}\sum_{i=0}^{3}\Delta_ig^{-i}+\frac{1}{M_2}\sum_{i=0}^{6}C_ig^{-i}+O(M^{-3})\right] \qquad (4.12)$$

where

$$\Delta_0=\frac{1}{p}s_1^2-s_2+2(\sigma_1-\alpha\sigma_0) \qquad (4.13)$$

$$\Delta_1=-\Delta_0-2\sigma_1+\frac{2\sigma_0}{p}(s_1+\alpha p) \qquad (4.14)$$

$$\Delta_2=-\Delta_0-\Delta_1+\frac{1}{p}\sigma_0^2 \qquad (4.15)$$

$$\Delta_3=-\frac{1}{p}\sigma_0^2 \qquad (4.16)$$

$$C_0 = \frac{1}{2}\Delta_0^2 + 2\alpha\Delta_0 + 4\alpha^2\sigma_0 + \frac{4}{3}\left(\frac{s_1^3}{p^2} - s_3\right) - \frac{\gamma}{p^2} \tag{4.17}$$

$$\begin{aligned} C_1 =& \frac{2s_1}{p}(\Delta_0 + \Delta_1 + 2\sigma_1) - \frac{4}{p}s_1\Delta_0 - \frac{(2+4\alpha p)}{p}\Delta_0 + \Delta_0\Delta_1 - 2\alpha(\Delta_0 + \Delta_1 + 2\sigma_1) \\ &+ \frac{4}{p}(\sigma_1 - \alpha\sigma_0) - 8\alpha\sigma_1 + 4\sigma_2 \end{aligned} \tag{4.18}$$

$$\begin{aligned} C_2 =& 4(\alpha\sigma_1 - 2\sigma_2) + \frac{1}{p}\sigma_0^2\Delta_0 - \Delta_0(\Delta_0 + \Delta_1) + \frac{1}{2}\Delta_1^2 + \frac{2\sigma_0}{p}(\Delta_0 + \Delta_1 + 2\sigma_1) - \frac{2\sigma}{p}\sigma_0^2 \\ &- \frac{4s_1}{p}(\Delta_0 + \Delta_1 + 2\sigma_1) - \frac{2}{p}(\Delta_0 + \Delta_1 + 2\sigma_1) + \frac{4\alpha}{p}\sigma_0 + \frac{\gamma}{p^2} - \frac{4}{3}\left(\frac{s_1^3}{p^2} - s_3\right) \\ &- \frac{4\sigma_0}{p}\Delta_0 + \frac{4}{p}s_1\Delta_0 + \frac{2(1+\alpha p)}{p}\Delta_0 + \frac{4\sigma_0}{p}(2\sigma_1 - 2\alpha\sigma_0) + \frac{4}{p}\alpha\sigma_0 - 4\alpha^2\sigma_0 - \frac{4}{p}\sigma_1 \\ &+ \frac{8\sigma_1}{p}(s_1 + \alpha p) + 4\alpha\sigma_1 + \frac{4}{p}\{\sigma_0(\alpha\sigma_0 - 2\sigma_1) + \sigma_1\} \end{aligned} \tag{4.19}$$

$$\begin{aligned} C_3 =& -\frac{8\sigma_1}{p}(s_1 + \alpha p) - \frac{4}{p}(\sigma_1 - 2\sigma_0\sigma_1) + \frac{4}{3p^2}\sigma_0^3 - \frac{2\sigma_0^2}{p^2} - \frac{4\alpha\sigma_0}{p} \\ &+ \frac{1}{p}\{4p\sigma_2 - 2\sigma_0^2\sigma_1 + 8\alpha\sigma_0^2 + (\sigma_0^2 - 6\sigma_0)(\Delta_0 + \Delta_1 + 2\sigma_1)\} \\ &+ \frac{2(s_1+\alpha p+1)}{p}(\Delta_0+\Delta_1+2\sigma_1) - \frac{2}{p}\sigma_0^2\Delta_0 - \Delta_1(\Delta_0+\Delta_1) + \frac{4\sigma_0}{p}\Delta_0 \end{aligned} \tag{4.20}$$

$$C_4 = \frac{1}{2}(\Delta_0+\Delta_1)^2 + \frac{4\sigma_0}{p}(\Delta_0+\Delta_1) - \frac{2\sigma_0^2}{p}\left(\frac{1}{2}\Delta_0+\Delta_1\right) + \frac{2(1-\alpha p)}{p^2}\sigma_0^2 - \frac{4\sigma_0^3}{p^2} + \frac{1}{2p^2}\sigma_0^4 \tag{4.21}$$

$$C_5 = -\frac{\sigma_0^4}{p^2} + \frac{8}{3}\frac{\sigma_0^3}{p^2} + \frac{\sigma_0^2}{p}(\Delta_0 + \Delta_1) \tag{4.22}$$

$$C_6 = \frac{1}{2p^2}\sigma_0^4. \tag{4.23}$$

Now inverting (4.12), we get the following result.

Theorem 4.1. *Under the sequence of local alternatives given by (4.3), the distribution function of $-2\rho\ell n\Lambda$ can be expanded as*

$$\begin{aligned} P[-2\rho\ell n\Lambda \le w] =& P[\chi_v^2(2\sigma_0) \le w] + \frac{1}{M}\sum_{i=0}^{3}\Delta_i P[\chi_{v+2i}^2(2\sigma_0) \le w] \\ &+ \frac{1}{M^2}\sum_{i=0}^{6} C_i P[\chi_{v+2i}^2(2\sigma_0) \le w] + O(M^{-3}) \end{aligned} \tag{4.24}$$

where the random variable $\chi_v^2(\sigma_0)$ is distributed as a non-central chi-square with v d.f. and non-centrality parameter σ_0, and the constants $\Delta_i's$ and $C_i's$ are given by (4.13)-(4.16) *and* (4.17)-(4.23) *respectively.*

Substituting $\mu = \mu_0$ in the coefficients $\Delta_i's$ and $C_i's$ in (4.24), we get the following corollary.

Corollary 4.1. *Under the sequence of local alternatives $A_N : \mu = \mu_0$; $I - q\Sigma^{-1} = \frac{2}{M}P$, the distribution function of $-2\rho\ell n\Lambda$ can be expanded as*

$$P[-2\rho\ell n\Lambda \le w] = P[\chi_v^2 \le w] + \frac{\Delta_0}{M}\{P[\chi_v^2 \le w] - P[\chi_{v+2}^2 \le w]\} + \frac{1}{M^2}\sum_{i=0}^{2} C_i P[\chi_{v+2i}^2 \le w] + O(M^{-3}) \tag{4.25}$$

where $\Delta_0 = \frac{1}{p}s_1^2 - s_2$,

$$C_0 = \frac{1}{2}\Delta_0^2 + 2\alpha\Delta_0 + \frac{4}{3}\left(\frac{s_1^2}{p^2} - s_3\right) - \frac{\gamma}{p^2}$$

$$C_1 = -\Delta_0^2 - \frac{4}{p}s_1\Delta_0 - \frac{(2+4\alpha p)}{p}\Delta_0$$

and $C_2 = -(C_0 + C_1)$.

Now consider another alternative hypothesis

$$A_N : q^{-1/2}(\mu - \mu_0) = \left(\frac{2}{M}\right)^{1/2}\delta; \quad I - q^{-1}\Sigma = \frac{2Q}{M} \tag{4.26}$$

where Q is a fixed matrix as $M \to \infty$. The asymptotic nonnull distribution of $-2\rho\ell n\Lambda$ for this case is obtained by making the following replacements in the coefficients $\Delta_i's$ and $C_i's$ in (4.24).

$$\text{tr } P \to -\text{tr } Q - \frac{2}{M}\text{tr } Q^2 = -r_1 - \frac{2}{M}r_2$$

$$\text{tr } P^2 \to \text{tr } Q^2 + \frac{4}{M}\text{tr } Q^3 = r_2 + \frac{4}{M}r_3$$

$$\text{tr } P^3 \to -\text{tr } Q^3 = -r_3$$

$$\sigma_1 \to -\delta' Q\delta - \frac{2}{M}\delta Q^2\delta = -\sigma_1^* - \frac{2}{M}\sigma_2^*$$

$$\sigma_2 \to \delta' Q^2\delta = \sigma_2^*.$$

Theorem 4.2. *Under the sequence of local alternatives given by (4.26), the distribution function of* $-2\rho\,\ell n\Lambda$ *can be expanded as*

$$P[-2\rho\ell n\Lambda \le w] = P[\chi_v^2(2\sigma_0) \le w] + \frac{1}{M}\sum_{i=0}^{3}\Delta_i^* P[\chi_{v+2i}^2(2\sigma_0) \le w] + \frac{1}{M^2}\sum_{i=0}^{6} C_i^* P[\chi_{v+2i}^2(2\sigma_0) \le w + O(M^{-3}) \tag{4.27}$$

where the constants $\Delta_i^{*\prime}s$ *and* $C_i^{*\prime}s$ *are given below.*

$$\Delta_0^* = \frac{1}{p}r_1^2 - r_2 - 2(\sigma_1^* + \alpha\sigma_0), \qquad \Delta_1^* = -\Delta_0^* + 2\sigma_1^* + \frac{2\sigma_0}{p}(-r_1 + \alpha p)$$

$$\Delta_2^* = -\Delta_0^* - \Delta_1^* + \frac{1}{p}\sigma_0^2, \qquad \Delta_3^* = -\frac{1}{p}\sigma_0^2$$

$$C_0^* = \frac{1}{2}\Delta_0^{*2} + 2\alpha\Delta_0^* + 4\alpha^2\sigma_0 - \frac{4}{3}\Big(\frac{r_1^3}{p^2} - r_3\Big) - \frac{\gamma}{p^2} + 4\Big(\frac{r_1r_2}{p} - r_3 - \sigma_2^*\Big)$$

$$\begin{aligned} C_1^* = &-\frac{2r_1}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) + \frac{4}{p}r_1\Delta_0^* - \frac{(2+4\alpha p)}{p}\Delta_0^* + \Delta_0^*\Delta_1^* - 2\alpha(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) \\ &- \frac{4}{p}(\sigma_1^* + \alpha\sigma_0) + 8\alpha\sigma_1^* + 4\sigma_2^* - 4\Big(\frac{r_1r_2}{p} - r_3 - 2\sigma_2^* + \frac{\sigma_0 r_2}{p}\Big) \end{aligned}$$

$$\begin{aligned} C_2^* = &-4(\alpha\sigma_1^* + 2\sigma_2^*) + \frac{1}{p}\sigma_0^2\Delta_0^* - \Delta_0^*(\Delta_0^* + \Delta_1^*) + \frac{1}{2}\Delta_1^{*2} + \frac{2\sigma_0}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) \\ &- \frac{2\alpha}{p}\sigma_0^2 + \frac{4r_1}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) - \frac{2}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) + \frac{4\alpha}{p}\sigma_0 + \frac{\gamma}{p^2} \\ &+ \frac{4}{3}\Big(\frac{r_1^3}{p^2} - r_3\Big) - \frac{4\sigma_0}{p}\Delta_0^* - \frac{4}{p}r_1\Delta_0^* + \frac{2(1+\alpha p)}{p}\Delta_0^* - \frac{4\sigma_0}{p}(2\sigma_1^* + 2\alpha\sigma_0) \\ &+ \frac{4}{p}\alpha\sigma_0 - 4\alpha^2\sigma_0 + \frac{4}{p}\sigma_1^* - \frac{8\sigma_1^*}{p}(-r_1 + \alpha p) - 4\sigma_1^* + \frac{4}{p}\{\sigma_0(\alpha\sigma_0 + 2\sigma_1^*) - \sigma_1^*\} \\ &- 4\sigma_2^* + \frac{4\sigma_0 r_2}{p} \end{aligned}$$

$$\begin{aligned} C_3^* = &\frac{8\sigma_1^*}{p}(-r_1 + \alpha p) + \frac{4}{p}(\sigma_1^* - 2\sigma_0\sigma_1^*) + \frac{4}{3p^2}\sigma_0^3 - \frac{2\sigma_0^2}{p^2} - \frac{4\alpha\sigma_0}{p} + \frac{2\sigma_0^2\sigma_1^*}{p} + 4\sigma_2^* \\ &+ \frac{8\alpha\sigma_0^2}{p} + \frac{\sigma_0^2}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) - \frac{6\sigma_0}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) \\ &+ \frac{2(1 + \alpha p - r_1)}{p}(\Delta_0^* + \Delta_1^* - 2\sigma_1^*) - \frac{2}{p}\sigma_0^2\Delta_0^* - \Delta_1^*(\Delta_1^* + \Delta_0^*) + \frac{4\sigma_0}{p}\Delta_0^* \end{aligned}$$

$$C_4^* = \frac{1}{2}(\Delta_0^* + \Delta_1^*)^2 + \frac{4\sigma_0}{p}(\Delta_0^* + \Delta_1^*) - \frac{2\sigma_0^2}{p}\Big(\frac{1}{2}\Delta_0^* + \Delta_1^*\Big) + \frac{2(1-\alpha p)}{p^2}\sigma_0^2 - \frac{4\sigma_0^3}{p^2} + \frac{1}{2p^2}\sigma_0^4$$

$$C_5^* = -\frac{\sigma_0^4}{p^2} + \frac{8}{3}\frac{\sigma_0^3}{p^2} + \frac{\sigma_0^2}{p}(\Delta_0^* + \Delta_1^*) \text{ and } C_6^* = \frac{1}{2p^2}\sigma_0^4.$$

Substituting $\mu = \mu_0$ in the coefficients in (4.27) we get the following result.

Corollary 4.2. *Under the sequence of local alternatives $A_N : \mu = \mu_0;\ I - q^{-1}\Sigma = \frac{2}{M}Q$, the distribution function of $-2\rho\ell n\Lambda$ can be expanded as*

$$P[-2\rho\ \ell n\ \Lambda \le w] = P[\chi_v^2 \le w] + \frac{\Delta_0^*}{M}\{P[\chi_v^2 \le w] - P[\chi_{v+2}^2 \le w]\} + \frac{1}{M^2}\sum_{i=0}^{2} C_i^* P[\chi_{v+2i}^2 \le w] + O(M^{-3})$$

where $\Delta_0^* = \frac{1}{p}r_1^2 - r_2$

$$C_0^* = \frac{1}{2}\Delta_0^{*2} + 2\alpha\Delta_0^* - \frac{4}{3}\Big(\frac{r_1^3}{p^2} - r_3\Big) - \frac{\gamma}{p^2} + 4\Big(\frac{r_1r_2}{p} - r_3\Big)$$

$$C_1^* = \frac{4}{p}r_1\Delta_0^* - \frac{(2+4\alpha p)}{p}\Delta_0^* - \Delta_0^{*2} - 4\Big(\frac{r_1r_2}{p} - r_3\Big)$$

and $C_2^* = (C_1^* + C_0^*)$.

For $\mu = \mu_0$; $\Sigma = \sigma^2 I$, we get the following corollary of the above theorem.

Corollary 4.3. *Under the null hypothesis $H : \mu = \mu_0;\ \Sigma = \sigma^2 I$, the distribution function of $-2\rho\ell n\Lambda$ can be expanded as*

$$P[-2\rho\ \ell n\Lambda \le w] = P[\chi_v^2 \le w] + \frac{\gamma}{M^2p^2}\{P[\chi_{v+4}^2 \le w] - P[\chi_v^2 \le w]\} + O(M^{-3}).$$

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